

EQUIVARIANT K -THEORY OF SMOOTH PROJECTIVE SPHERICAL VARIETIES

SOUMYA BANERJEE AND MAHIR BILEN CAN

ABSTRACT. We present a description of the equivariant K -theory of a smooth projective spherical variety. This provides an integral K -theory version of Brion’s calculation of equivariant Chow-cohomology of such varieties. We consider the equivariant K -theory of wonderful compactifications of minimal rank symmetric varieties. We obtain a formula for their structure constants in terms of certain lower dimensional Schubert classes. This generalizes results of Uma on equivariant compactifications of adjoint groups.

1. INTRODUCTION

The foundations of equivariant algebraic geometry have matured enough to support a wealth of results that match their topological counterparts. These results pave the way for studying equivariant (generalized) cohomology theories of algebraic varieties with linear algebraic group actions. A broad class of such varieties: toric varieties, and spherical varieties also happen to admit concrete combinatorial descriptions. This leads to a rich interaction between geometry, and combinatorics. In this paper we study the equivariant K -theory of smooth projective spherical varieties with a particular emphasis on the wonderful compactifications of minimal-rank symmetric varieties.

1.0.1. Building on the earlier ideas of Chang and Skjelbred, in their influential paper [GKM98], Goresky, Kottwitz and MacPherson introduced an effective method to calculate topological equivariant cohomology of an equivariantly formal space. The subsequent applications and refinements of their methods are now commonly referred to as “GKM-type” results. In algebraic-geometry Brion obtained the first GKM-type presentation of equivariant rational Chow-ring of a smooth spherical variety, see [Bri97], using the equivariant intersection theory of Edidin and Graham.

The study of equivariant algebraic K -theory for coherent sheaves was initiated in early 1980’s by Thomason [Tho87]¹. Almost three decades after its inception, Vezzosi and Vistoli, in [VV02, VV03], substantially extended Thomason’s work, which paves the way for GKM-type results in algebraic K -theory.

BEN-GURION UNIVERSITY OF THE NEGEV, ISRAEL

TULANE UNIVERSITY, NEW ORLEANS, USA

E-mail addresses: soumya@cs.bgu.ac.il, mcan@tulane.edu.

Date: February 11, 2017.

2010 *Mathematics Subject Classification.* 19E08, 14M27.

Key words and phrases. Equivariant K -theory, equivariant Riemann-Roch, spherical varieties, minimal rank symmetric varieties, wonderful compactification.

¹Note that in literature one denotes the Quillen K -functors on coherent sheaves by G and on vector bundles by K . In this paper, we work with coherent sheaves and use K unambiguously.

As the first application of Vezzosi and Vistoli results, Uma calculated presentations of the equivariant Grothendieck K -groups of the wonderful compactification of a semisimple reductive group of adjoint type, see [Uma07].

Working on a slightly different question, Joshua and Krishna, in [JK15], show that for a smooth projective spherical G -variety X (defined over a field k), with a maximal torus $T \subset G$, there is a ring isomorphism

$$K_{T,0}(X) \otimes_{\mathbb{Z}} K_{T,*}(k) \cong K_{T,*}(X).$$

This result together with Uma's work recovers the complete equivariant K -theory in the group case. One of the goals of our paper is to extend these ideas to other smooth and projective spherical varieties.

Recently, Anderson and Payne, in [AP15], initiated a study of a operational bivariant theory associated to the Grothendieck groups of coherent equivariant sheaves. Using this operational K -theory, Gonzalez extended Uma's results to possibly non-smooth spherical G -varieties admitting finitely many torus invariant curves (also called T -skeletal varieties); see [Gon15]. These results provide answers for certain singular varieties, however at present they are not applicable to non- T -skeletal cases. Our work, on the other hand, applies to all smooth projective spherical varieties.

1.0.2. This paper was motivated by a question of Dan Edidin who drew our attention to extending the equivariant K -theory computations in the group case to other spherical varieties. Our first result is an integral K -theoretic analogue of Brion's presentation of equivariant Chow-groups for spherical varieties in [Bri97, Theorem 7.3].

Theorem 1.1. *Let X be a smooth projective spherical G variety. Let T be a maximal torus of G , Φ_G the roots of G with respect to T , $R(T)$ the representation ring of T and W the Weyl group of G . The T -equivariant K -theory $K_{T,*}(X)$ is the ring of ordered tuples $(f_x) \in \prod_{x \in X^T} K_*(k) \otimes R(T)$ satisfying the following congruence conditions:*

- (1) $f_x - f_y = 0 \pmod{(1 - \chi)}$ when x, y are connected by a T -stable curve with weight χ .
- (2) $f_x - f_y = f_x - f_z = 0 \pmod{(1 - \chi)}$ and $f_y - f_z = 0 \pmod{(1 - \chi^2)}$, where $\chi \in \Phi_G$, and an irreducible component of the subvariety $X^{\text{Ker}(\chi)}$ which contains the points x, y, z is isomorphic to \mathbb{P}^2 . There is an element in W that fixes x and permutes the point y and z .
- (3) $f_x - f_y = f_y - f_z = f_z - f_w = f_x - f_w = 0 \pmod{(1 - \chi)}$, where $\chi \in \Phi_G$, and an irreducible component of the subvariety $X^{\text{Ker}(\chi)}$ which contains the points x, y, z, w is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. There is an element in W that fixes two points and permutes the other two.
- (4) $f_x - f_y = f_z - f_w = 0 \pmod{(1 - \chi)}$ and $f_y - f_z = 0 \pmod{(1 - \chi^{2n})}$ and $f_x - f_w = 0 \pmod{(1 - \chi^n)}$, where $n > 1$, and $\chi \in \Phi_G$. An irreducible component of the subvariety $X^{\text{Ker}(\chi)}$ which contains the points x, y, z, w is isomorphic to the ruled surface \mathbb{F}_n . There is an element in W that fixes the points x and w and permutes z and y .

The Weyl group W acts on the torus fixed point set X^T by permutation and the G -equivariant K -theory is obtained by taking W -invariants.

The natural strategy to prove this result is to reduce the problem to the computation of some small dimensional (specifically rank one spherical SL_2 compactifications) spherical varieties. The classification of such varieties is well known due to Ahiezer (see [Ahi83, Bri86]).

The original problem is now tantamount to calculating the equivariant K -theory of finitely many small dimensional varieties. At this point one encounters the issue that not all spherical varieties are T -skeletal and can have positive dimensional families of torus invariant curves which must be accounted for. A good example is the classical variety of complete quadrics. We circumvent this issue by rigidifying the problem using toric geometry. We recover the necessary K -theory from the toric case by using a co-base change theorem, see Proposition 2.7.

When a spherical variety X has finitely many T -invariant curves, GKM-type theorems are much more tractable and often contain additional combinatorial structures. The T -equivariant K -theory of flag varieties is perhaps the most well studied example, see [KK90]. In this vein, a natural family of varieties are the wonderful compactifications, due to De Concini and Procesi (see [DCP83]), of the minimal-rank² symmetric spaces. The work of Tchoudjem, in [Tch07], gives a combinatorial description of the torus stable points and curves in the wonderful compactifications of minimal rank symmetric varieties. When G is a semisimple group of adjoint type a complete classification of the irreducible components up-to isomorphism is known. These are the wonderful compactifications of three families (i) PSL_{2n}/PSp_n , (ii) PSO_{2n}/PSO_{2n-1} , (iii) The group case : $G \simeq G \times G/\Delta G$ (where ΔG is the diagonal embedding) and an isolated exceptional case E_6/F_4 . Many important results, in the study of equivariant cohomology, were obtained in the group case by Bifet, De Concini and Procesi in their seminal work in [BDCP90], and more recently, their has been extended by Strickland in [Str12].

In the K -theory setting, the group case was thoroughly investigated bu Uma in [Uma07]. Building up on Tchoudjem's work we are able to generalize Uma's result to wonderful compactifications of all families of minimal-rank symmetric spaces.

Theorem 1.2. *[See Proposition 5.9 for details] Let X denote the wonderful compactification of an irreducible minimal rank symmetric variety G/H . Then $K_{G,*}(X)$ has a decomposition of the form*

$$K_{G,*}(X) = K_{S,*}(Y_0) \otimes R(T/S)^{W_H},$$

where Y_0 is an affine toric variety, W_H is the Weyl group of H and S is a maximal anisotropic subtorus of T .

An immediate consequence of this theorem is that the G -equivariant K -theories of wonderful compactifications of $PSL_n \times PSL_n/\Delta PSL_n$ and PSL_{2n}/PSp_n are identical.

There is an important basis of the torus equivariant K -theory of flag variety, the Schubert basis, which has deep combinatorial structure. In the minimal rank case we show that there is a basis which enjoys the same combinatorial properties as the Schubert basis and the sturcture constants, in this basis, are related to that of an appropriate Schubert basis.

Outline of the paper. We will present a brief outline of the contents of this paper.

In Section 2, we collect several results in equivariant K -theory that are crucial to the rest of the paper. Some of these results are well known and some are new.

In Section 3, we study the geometric structure of the fixed point locus X^S , where $S \subset T$ is a codimension one subtorus. This is an important step in reduction step that goes into the proof of Theorem 1.1.

²The minimal rank condition ensures that such varieties have finitely many torus fixed curves.

In Section 4, we combine the results of the previous sections to prove Theorem 1.1. We start with explicit presentation in the rank one case of the base cases and then bootstrap it to the general case.

In Section 5, we recall some relevant structure theory of wonderful compactifications of symmetric varieties of minimal rank. We present a proof of Theorem 1.2 and several variants that extend the results of Uma, in [Uma07]. In the short appendix, we use equivariant Riemann-Roch theorem to relate our work with Brion's Chow-ring computations.

Acknowledgments. The present paper owes its existence to Dan Edidin who drew our attention to this question and generously answered our technical questions. We are extremely grateful to Michel Brion for his invaluable guidance, comments, corrections and encouragement in the final stages of this work.

We thank Dave Anderson, Mikhail Kapranov, Sam Payne and Lex Renner for their crucial remarks and suggestions during various stages of this work. The first author would like to thank Amnon Besser, Ilya Tyomkin and Amnon Yekutieli for suggestions and answers to technical questions.

1.0.3. Assumptions/ Notations. A variety is a reduced scheme of finite type defined over an algebraically closed field k of characteristic zero. It is allowed to have irreducible components. Points of a variety will always mean closed points. The representable functor, from schemes over k to the category of sets, associated to a scheme X will be denoted by \underline{X} . The set $\underline{X}(A)$ will denote the A valued points of X for any k -algebra A .

Unless otherwise stated, the linear group G is always connected and reductive. A G -variety X is a normal variety with an algebraic action of G . Given any closed subgroup $H \subset G$ the neutral component of H will mean the component of H containing the identity. The semisimple part of a reductive group G will be denoted by G^{ss} .

We recall that *rank* has two, potentially confusing, meanings. The rank of a linear groups is its semisimple-rank where as the rank of a spherical variety of homogeneous space is defined in Section 3.1.

All tensor products will be over \mathbb{Z} unless specified.

2. RESULTS IN K -THEORY

In this paper the term *equivariant algebraic K -theory* will mean the algebraic K -theory of the category of equivariant coherent sheaves on a G -variety X . The foundational results were obtained by Thomason; see [Tho87]. Informally speaking, in this theory, one applies Quillen's Q -construction to the abelian category of equivariant sheaves.

Definition 2.1 (Equivariant sheaf). Let a denote the action map $a : G \times X \rightarrow X$ and let $p_X : G \times X \rightarrow X$ denote the second projection. A G -equivariant sheaf on X is a pair (\mathcal{F}, ϕ) where \mathcal{F} is a coherent sheaf on X and ϕ is an isomorphism $\phi : a^*\mathcal{F} \rightarrow p_X^*\mathcal{F}$ of sheaves on $G \times X$, which satisfies a natural cocycle condition on $G \times G \times X \rightrightarrows G \times X$.

A morphism of equivariant sheaves, (\mathcal{F}, ϕ) and (\mathcal{F}', ϕ') , is a morphism of sheaves which commutes with isomorphisms ϕ and ϕ' . We denote the category of G -equivariant sheaves on X by $\text{Sh}_G(X)$. The equivariant K -groups, denoted by $K_{G,*}(X)$ are defined as the homotopy groups of the loop-space of the nerve of $Q\text{Sh}_G(X)$ (where $Q\text{Sh}_G(X)$ is the Quillen Q

construction applied to $\mathrm{Sh}_G(X)$). One can similarly define $K_{H,*}(X)$ for any closed subgroup $H \subset G$.

These K -groups admit several homomorphisms resulting functorially from equivariant maps between spaces and homomorphisms of the structure group G acting compatibly on a fixed space. As we will see, this is often very useful for calculations. We start with a general result that relates, for a group G and a closed subgroup H , the G -equivariant K -theory to H -equivariant K -theory.

Proposition 2.2 (Faddeev-Shapiro Lemma, [Mer97]). *Let H be a closed subgroup of the algebraic group G . Then for any G -variety X , the inclusion map $X \hookrightarrow X \times G/H$ defined by $x \mapsto (x, eH)$ induces the isomorphisms*

$$K_{G,n}(X \times G/H) \simeq K_{H,n}(X),$$

for all $n \geq 0$.

Notice that we do not assume G is reductive in the above proposition. When G is a reductive group and $B \subset G$ is a Borel subgroup there is an useful refinement.

Proposition 2.3 (Merkurjev, [Mer97]). *Let G be a (split) reductive group and B a Borel subgroup and X be a smooth projective G -variety then the natural map*

$$\theta : R(B) \otimes_{R(G)} K_{G,n}(X) \rightarrow K_{B,n}(X)$$

is an isomorphism for all $n \geq 0$.

Using the structure theory of solvable groups, we can decompose the Borel group B into a maximal torus T and a unipotent subgroup U . As an algebraic variety U is isomorphic to an affine space. The homotopy-invariance property of equivariant K -theory then implies $K_{B,n}(X) \simeq K_{B,n}(X \times B/T) \simeq K_{T,n}(X)$; see [Tho87, Theorem 4.1] for details.

Remark 2.4. In particular, we have

$$K_T(X) = R(T) \otimes_{R(G)} K_G(X).$$

When G is simply connected, a result of Steinberg (see [Ste75]) shows that

$$R(G) \simeq R(T)^W \text{ and } R(T) \simeq R(G) \otimes \mathbb{Z}[W],$$

where W is the Weyl group of (G, T) . The first isomorphism is an isomorphism of rings and the latter is only an isomorphism of $R(G)$ -modules with a compatible action of W . Consequently, we recover $K_G(X)$ as the space of W -invariants in $K_T(X)$.

In the next Proposition we consider a refinement of Proposition 2.2 for torus actions.

Proposition 2.5. *Let T be an algebraic torus and $T' \subset T$ is a fixed codimension one subtorus. Let X be a projective T -variety. Then the canonical map*

$$R(T') \otimes_{R(T)} K_{T,n}(X) \rightarrow K_{T',n}(X)$$

is an isomorphism.

Proof. Let \mathcal{L} be a one-dimensional representation of T such that $\mathcal{L} \setminus \{0\} = T/T'$. Let j denote the obvious map $j : X \times T/T' \hookrightarrow X \times \mathcal{L}$, and $p : X \times \mathcal{L} \rightarrow X$ denote the projection map. The pullback map $p^* : K_{T,n}(X) \rightarrow K_{T,n}(X \times \mathcal{L})$ is an isomorphism by the homotopy

invariance property. Let $\text{res} : \text{Sh}_T(X) \rightarrow \text{Sh}_{T'}(X)$ denote the canonical restriction map. In the commutative triangle in eqn.(1) the vertical equality is a consequence of Proposition 2.2.

$$(1) \quad \begin{array}{ccc} K_{T,n}(X) & \xrightarrow{(pj)^*} & K_{T,n}(X \times T/T') \\ & \searrow \text{res} & \parallel \\ & & K_{T',n}(X). \end{array}$$

Now consider the decomposition of the total space $X \times \mathcal{L}$

$$X = X \times \{0\} \xhookrightarrow{i} X \times \mathcal{L} \xleftarrow{j} X \times T/T'$$

into a closed set $X \times \{0\}$ and its open complement. The terms of the localization long-exact sequence in equivariant K -theory fit into the commutative diagram below.

$$(2) \quad \begin{array}{ccccccc} K_{T,n}(X) & \xrightarrow{i_* p^{*-1}} & K_{T,n}(X) & \xrightarrow{\text{res}} & K_{T',n}(X) \\ \parallel & & \downarrow p^* & & \parallel \\ \dots & \longrightarrow & K_{T,n}(X) & \xrightarrow{i_*} & K_{T,n}(X \times \mathcal{L}) & \xrightarrow{j^*} & K_{T,n}(X \times T/T') \longrightarrow \dots \end{array}$$

Thanks to [FG05, V, Corollary 27], the top horizontal row in Diagram (2) is split exact and the kernel of res is generated by the ideal $(1 - \chi)$ in $K_{T,n}(X)$; where χ is defined by the short exact sequence

$$1 \rightarrow T' \rightarrow T \xrightarrow{\chi} \mathbb{G}_m \rightarrow 1.$$

As a result, we have

$$K_{T',n}(X) = K_{T,n}(X)/(1 - \chi)K_{T,n}(X) = R(T') \otimes_{R(T)} K_{T,n}(X).$$

□

Corollary 2.6. *Let T be an algebraic torus and T' is any subtorus. Let X be any projective T -variety then the canonical map $R(T') \otimes_{R(T)} K_{T,n}(X) \rightarrow K_{T',n}(X)$ is an isomorphism.*

Sketch of the proof. We use induction on the codimension of T' in the torus T . We can and choose a chain of subtori

$$T' = T_0 \subset T_1 \subset \dots \subset T_r = T$$

such that each T_i is codimension one in T_{i+1} . The assertion follows from using Proposition 2.5 inductively at each step. □

Proposition 2.7. *Consider a fixed subtorus $T' \subset T$. Let X be a T -variety such that T' acts trivially on X . Then we have the following formula*

$$K_{T,*}(X) \cong R(T) \otimes_{R(T/T')} K_{T/T',*}(X).$$

Proof. The torus T/T' acts on X and the groups $K_{T/T',*}(X)$ are defined. Consider the following diagram

$$\begin{array}{ccc} & T/T' \times X & \\ \pi \nearrow & \downarrow \bar{a} \downarrow \overline{pr}_X & \\ T \times X & \xrightarrow[a]{pr_X} & X \end{array}$$

where a is the action map and \bar{a} is the induced action map. Let pr_T (resp. pr_X) denote the projection map from $T \times X$ to T (resp. X) and \overline{pr}_X denotes the projection to X . We will identify the isomorphisms $a^* \cong \pi^* \bar{a}^*$ and $pr_X^* \cong \pi^* \overline{pr}_X^*$ on sheaves.

Let χ be any character of T' . We fix an isomorphism $\phi : T' \times T/T' \rightarrow T$ throughout which induces the given embedding $T' \subset T$. Let $\hat{\chi}$ denote the unique character of T which restricts to χ on T' is trivial on T/T' (via ϕ). We introduce the notion of “ χ -twist” of an equivariant sheaf $(\mathcal{F}, \mu) \in \text{Sh}_{T/T'}(X)$ (which depends on $\hat{\chi}$). To this end, consider the map

$$(3) \quad \chi \cdot \mu := pr_T^*(\hat{\chi}) \cdot \pi^*(\mu) : a^*(\mathcal{F}) \rightarrow pr_X^*(\mathcal{F})$$

between sheaves $a^*(\mathcal{F})$ and $pr_X^*(\mathcal{F})$ on $T \times X$. Given any character η of T the regular function $pr_T^*(\eta)$ is an invertible function on $T \times X$. So the isomorphism $\chi \cdot \mu$ in the above equation (3) is an isomorphism of sheaves and $(\mathcal{F}, \chi \cdot \mu)$ is a T -equivariant sheaf on X .

The following is immediate from the definitions.

$$(4) \quad \text{Hom}_{\text{Sh}_T(X)}((\mathcal{F}, \chi \cdot \mu), (\mathcal{G}, \chi' \cdot \lambda)) = \begin{cases} 0 & \text{if } \chi \neq \chi' \\ \text{Hom}_{\text{Sh}_{T/T'}(X)}((\mathcal{F}, \mu), (\mathcal{G}, \lambda)) & \text{if } \chi = \chi' \end{cases}$$

We have thus constructed a functor $\mathcal{T}_\chi : \text{Sh}_{T/T'}(X) \rightarrow \text{Sh}_T(X)$ which is full and faithful. Given any object \mathcal{F} of $\text{Sh}_{T/T'}(X)$ we call $\mathcal{T}_\chi(\mathcal{F})$ the χ -twist of \mathcal{F} and let $\text{Sh}_T(X)_\chi$ denote the subcategory $\mathcal{T}_\chi(\text{Sh}_{T/T'}(X))$ of $\text{Sh}_T(X)$ and for any character χ' of T' (also extended to T via ϕ) the functor $\mathcal{T}_{\chi'}$ defines equivalence of categories

$$(5) \quad \mathcal{T}_{\chi'} : \text{Sh}_T(X)_\chi \xrightarrow{\cong} \text{Sh}_T(X)_{\chi' \cdot \chi} = \mathcal{T}_{\chi' \cdot \chi}(\text{Sh}_{T/T'}(X)).$$

The map of group schemes $T \rightarrow T/T'$ is faithfully flat, by [Wat79, Theorem 14.1], so the functor π^* induces an equivalence between the category of sheaves $\text{Sh}(X \times T/T')$ and $\text{Sh}(X \times T)$ (see [BLR12, Chapter 6]). Let \mathcal{D} denote the inverse equivalence. Then given any equivariant sheaf (\mathcal{F}, α) in $\text{Sh}_T(X)$ we get an isomorphism

$$\mathcal{D}(\alpha) : \bar{a}^*(\mathcal{F}) \rightarrow \overline{pr}_X^*(\mathcal{F})$$

of sheaves on $X \times T/T'$. This makes $(\mathcal{F}, \mathcal{D}(\alpha))$ into a T/T' -equivariant sheaf. The functor π^* also preserves equivariant sheaves. This follows from the observation that π^* is identical to the functor \mathcal{T}_{χ_e} , where χ_e is the trivial character.

Consider the abelian sub-category \mathcal{C} of $\text{Sh}_T(X)$ whose objects are T -equivariant sheaves $(\mathcal{F}, \chi \cdot \mathcal{D}(\alpha))$, for any character χ of T' , and morphisms

$$\text{Hom}_{\mathcal{C}}((\mathcal{F}, \chi \cdot \mathcal{D}(\alpha)), (\mathcal{G}, \chi' \cdot \mathcal{D}(\alpha))) := \text{Hom}_{\text{Sh}_T(X)}((\mathcal{F}, \chi \cdot \mathcal{D}(\alpha)), (\mathcal{G}, \chi' \cdot \mathcal{D}(\alpha)))$$

(see eqn. (4) above). Then \mathcal{C} is equivalent to $\text{Sh}_T(X)$ since any object (\mathcal{F}, μ) in $\text{Sh}_T(X)$ is of the form $(\mathcal{F}, \chi_e \cdot \mathcal{D}(\mu))$ and the Hom-sets in $\text{Sh}_T(X)$ are graded by the lattice of characters of T' . The K -theory of $\text{Sh}_T(X)$ is the same as the K -theory of \mathcal{C} .

It follows from Quillen’s Q-construction, that the space $BQ(\mathcal{C})$ is a disjoint discrete $\text{Hom}(T', \mathbb{G}_m)$ -fold covering space of $BQ(\text{Sh}_{T/T'}(X))$. The functors \mathcal{T}_χ induce an action of $R(T')$ on $BQ(\mathcal{C})$ (see eqn. 5). Calculating the homotopy groups we get

$$K_{T,*}(X) = K_{T/T',*}(X) \otimes_{K_*(k)} R(T').$$

The module $K_{T/T',*}(X)$ is an $R(T/T')$ module and $R(T) = R(T') \otimes R(T/T')$. This proves the assertion. \square

Remark 2.8. We note an alternate way to prove Proposition 2.7 without explicitly using the Q-construction: Theorem 1.2 of [JK15] reduces the original problem to understanding the relation between the Grothendieck K -groups $K_{T,0}(X)$ and $K_{T/T',0}(X)$. This follows from the constructions in the first part of the above proof.

2.1. Equivariant K -theory of toric varieties. We recall the fundamental result of Vezzosi and Vistoli which is at the foundation of most approaches to equivariant algebraic K -theory of G -varieties.

Theorem 2.9 (Vezzosi-Vistoli [VV03]). *Suppose G is a diagonalizable group acting on a smooth proper scheme X defined over a perfect field; denote by T the toral component of G , that is the maximal subtorus contained in G . Then the restriction homomorphism on K -groups $K_{G,*}(X) \rightarrow K_{G,*}(X^T)$ is injective, and its image equals the intersection of all images of the restriction homomorphisms $K_{G,*}(X^S) \rightarrow K_{G,*}(X^T)$ for all subtori $S \subset T$ of codimension 1.*

Among its many applications, the theorem provides a complete description of torus equivariant K -theory of smooth toric varieties. Toric varieties will play an important role in the following sections so we recall some basic results from the theory of toric varieties. We refer the reader to [Ful93] for details.

Let T be an algebraic torus and we denote its lattice of characters (respectively, co-characters) by M_T (respectively, by N_T). Recall that $R(T) = \mathbb{Z}[M_T]$ and $T = \text{Hom}_{gp}(M_T, \mathbb{G}_m)$. Given any homomorphism $\phi : T' \rightarrow T$ we have a map $\phi^* : \mathbb{Z}[M_T] \rightarrow \mathbb{Z}[M_{T'}]$ which makes $\mathbb{Z}[M_{T'}]$ a $\mathbb{Z}[M_T]$ module. When ϕ is a closed embedding the induced map $\phi^* : M_T \rightarrow M_{T'}$ is surjective. We associate the subgroup $M_{T'}^\perp \subset M_T$ which consists of all characters of T that are trivial on T' and we have a non-canonical splitting $M_T = M_{T'} \oplus M_{T'}^\perp$. When the underlying torus is clear from context we will drop the subscript T from the (co-)character lattices.

A rational fan Δ in $N_\mathbb{R} := N \otimes \mathbb{R}$ defines a toric variety, denoted by $X(\Delta)$, with a dense open set $T = \text{Hom}_{gp}(M, \mathbb{G}_m)$. We will exclusively work with rational fans and simply refer to them as fans. Let $\Delta_1 \subset \Delta$ denote the finite set of all one-dimensional cones of Δ . We restrict ourselves to toric varieties $X(\Delta)$ which are (i) smooth and (ii) projective; translated to the language of fans these conditions correspond to the restrictions (i) Δ_1 forms a lattice basis of N in the real vector space $N_\mathbb{R}$ and (ii) the support of the fan $|\Delta|$ is all of $N_\mathbb{R}$.

There is a bijection between cones of the fan Δ and the orbits of the torus T in $X(\Delta)$: the torus orbit $O_\sigma \subset X(\Delta)$, corresponding to a cone $\sigma \in \Delta$, is the set of all points in $X(\Delta)$ which are stabilized by the subtorus $\text{Hom}_{gp}(M/M(\sigma), \mathbb{G}_m)$ of T ; where $M(\sigma) := \sigma^\perp \cap M$ is the subspace of characters trivial on the cone σ . It turns out that the set O_σ is a Zariski open subset of its Zariski closure in $X(\Delta)$. This shows that, among other things, that $\dim_\mathbb{R}(O_\sigma) = \text{codim}_\mathbb{R}(\sigma)$. In particular cones of maximal dimension in Δ correspond to T -fixed points and the cones of codimension one correspond to T -stable curves. The identification of orbits and cones imply $O_\sigma = T/T_\sigma$ and hence

$$K_{T,*}(O_\sigma) = K_{T,*} \otimes R(T/T_\sigma) = K_*(k) \otimes R(T_\sigma) = K_*(k)[M(\sigma)].$$

In the light of Theorem 2.9, we conclude that natural restriction map

$$(6) \quad K_{T,*}(X(\Delta)) \hookrightarrow \prod_{\sigma \in \Delta_{\max}} K_*(k) \otimes R(T_\sigma)$$

where Δ_{\max} is the set of maximal cones in Δ is an injective map of $R(T)$ modules. Moreover the image is characterized by the collection of elements

$$(a_\sigma) \in \prod_{\sigma \in \Delta_{\max}} K_*(k) \otimes R(T_\sigma)$$

which satisfy: for any two maximal cones σ_1 and σ_2 of Δ the restrictions of a_{σ_1} and a_{σ_2} to $R(T_{\sigma_1 \cap \sigma_2})$ coincide.

Concretely, for any maximal cone σ of Δ let us identify T_σ with T , and $T_{\sigma_1 \cap \sigma_2}$ with the codimension one subtorus $\text{Ker}(\chi_{\sigma_1 \cap \sigma_2}) \subset T$, where $\chi_{\sigma_1 \cap \sigma_2}$ is the unique generator of $M_{T_{\sigma_1 \cap \sigma_2}}^\perp$. Then an element $(f_\sigma)_{\sigma \in \Delta_{\max}}$ in the right hand side of eqn.(6) belongs to $K_{T,*}(X(\Delta))$ if and only if it satisfies the condition

$$(7) \quad f_{\sigma_1} - f_{\sigma_2} = 0 \mod (1 - \chi_{\sigma_1 \cap \sigma_2})$$

for any pair of intersecting maximal cones σ_1 and σ_2 .

A presentation of K -theory in this form will be called *GKM* presentation.

Remark 2.10. We note in passing that one can consider a sheafified version of equivariant K -groups on toric varieties and use the techniques developed in [Tho85] to extend these results to general, not necessarily smooth or projective, toric varieties.

In particular one shows that for affine (even non-smooth!) toric varieties U_σ there is a formula

$$K_{T,*}(U_\sigma) = R(T_\sigma) \otimes K_*(k)$$

where T_σ is the stabilizer of a geometric point. For a general toric variety $X(\Delta)$ using the natural cover by affine opens sets $\{U_\sigma\}_{\sigma \in \Delta_{\max}}$ and the exactness of the complex

$$(8) \quad 0 \longrightarrow K_{T,*}(X) \longrightarrow \bigoplus_{\sigma \in \Delta_{\max}} K_{T,*}(U_\sigma) \xrightarrow{\partial} \bigoplus_{\sigma, \tau \in \Delta_{\max}, \sigma \cap \tau \neq \emptyset} K_{T,*}(U_{\sigma \cap \tau})$$

$$(f_\sigma)_{\sigma \in \Delta_{\max}} \xrightarrow{\partial} (f_\tau|_{U_{\sigma \cap \tau}} - f_\sigma|_{U_{\sigma \cap \tau}})_{\sigma, \tau \in \Delta_{\max}, \sigma \cap \tau \neq \emptyset}$$

one can generalize eqn.(7); see [AHW09] for details.

There is another description in-terms of generators and relations for the torus equivariant K -theory of smooth projective toric varieties $X(\Delta)$ called the *multiplicative Reisner-Stanley* (RS) presentation. These presentations were extensively studied in the context of equivariant cohomology of regular embeddings by Biffet, De Concini and Procesi in [BDP90].

Proposition 2.11 (The Reisner-Stanley presentation). *Let $X(\Delta)$ be a smooth toric variety. If Δ_1 denotes the set of all one dimensional cones of Δ , then there is an isomorphism of $K_*(k)$ -algebras*

$$(9) \quad i : \frac{K_*(k)[x_\rho^{\pm 1}]}{(\prod_{\rho \in S} (x_\rho - 1))} \simeq K_{T,*}(X(\Delta)),$$

where the product in the quotient is taken over all subsets $S \subseteq \Delta_1$ satisfying the condition:

$$(10) \quad \text{the elements of } S \text{ are not all contained in a maximal cone in } \Delta.$$

Sketch of the Proof. Given any one-dimensional cone $\rho \in \Delta_1$, let v_ρ denote the generator of the (rank-one) monoid $\mathbb{N} \cdot \rho \cap N$. Let us denote the one dimensional faces of a cone σ in the fan Δ by $\rho_{1,\sigma}, \rho_{2,\sigma}, \dots, \rho_{k,\sigma}$. The fact that $X(\Delta)$ is smooth and projective ensures that for any maximal cone σ the vectors $v_{\rho_{1,\sigma}}, v_{\rho_{2,\sigma}}, \dots, v_{\rho_{k,\sigma}}$ form an integral basis of N and the dual generators $v_{\rho_{i,\sigma}}^\vee$ form an integral basis of M .

Given any one-dimensional cone $\rho \in \Delta_1$ and any $\sigma \in \Delta_{\max}$ consider the assignment

$$u_\rho^\sigma := \begin{cases} 1 & \text{if } \rho \text{ is not a face of the cone } \sigma, \\ v_{\rho_{i,\sigma}}^\vee & \text{if } \rho = \rho_i \text{ is a face of the cone } \sigma \end{cases}$$

in $R(T) = \mathbb{Z}[M]$.

The map i in eqn.(9) is defined by mapping

$$(11) \quad x_\rho \mapsto (u_\rho^\sigma)_{\sigma \in \Delta_{\max}} \in \prod_{\sigma \in \Delta_{\max}} R(T_\sigma)$$

as ρ varies over all one dimensional cones of Δ .

We refer the reader to [VV03, Theorem 6.4] which shows that the assignment in eqn.(11) defines the correct image. \square

Remark 2.12. We point out that the presentation, given by eqn.(9), doesn't explicitly show the $R(T)$ -module structure on the left hand side. It can be recovered from the description of the map i given by (eqn.11) above on a case by case basis.

Next, we will consider four cases which will be used in the sequel.

2.2. Toric \mathbb{P}^1 . The fan of \mathbb{P}^1 as a toric variety is shown in Figure 1. In this case the maximal cells are one-dimensional. However, to be consistent with the notation used before, we continue to use σ_i to denote the maximal cones and ρ_i to denote the one dimensional rays. The underlying torus T in this case is one dimensional so $R(T) = \mathbb{Z}[\chi^\pm]$. We have the formula

$$\prod_{\sigma \in \Delta_{\max}} K_*(k) \otimes R(T_\sigma) = K_*(k)[\chi^\pm]_{\sigma_1} \times K_*(k)[\chi^\pm]_{\sigma_2}.$$

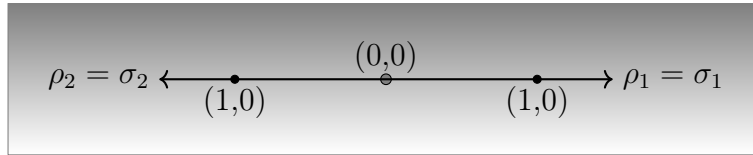


FIGURE 1. The fan of \mathbb{P}^1 .

GKM Presentation. We note that the intersection of the maximal cones $\sigma_1 \cap \sigma_2$ is the zero cone whose stabilizer is the trivial group. As a result, the GKM description of the K groups is

$$K_{T,*}(\mathbb{P}^1) = \left\{ (f_1, f_2) \in K_*(k)[\chi^\pm] \times K_*(k)[\chi^\pm] \mid \begin{array}{l} \text{the constant term of } f_1 \\ = \text{the constant term of } f_2 \end{array} \right\}.$$

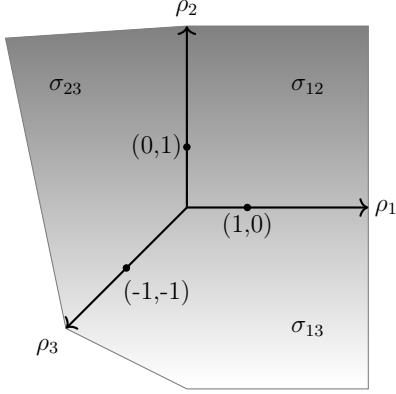


FIGURE 2. The fan of \mathbb{P}^2 .

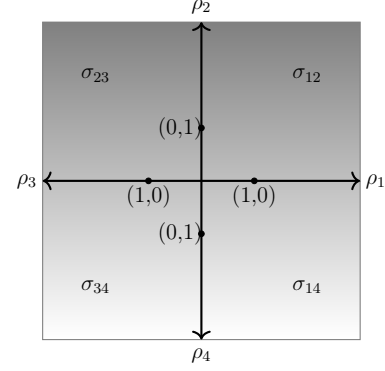


FIGURE 3. The fan of $\mathbb{P}^1 \times \mathbb{P}^1$.

RS Presentation. The Reisner-Stanley presentation is given by

$$\frac{K_*(k)[x_{\rho_1}^{\pm}, x_{\rho_2}^{\pm}]}{(x_{\rho_1} - 1)(x_{\rho_2} - 1)}.$$

Using Proposition 2.11 we see that the generators are mapped to

$$x_{\rho_1} \mapsto (\chi, 1) \text{ and } x_{\rho_2} \mapsto (1, \chi).$$

As a $K_*(k)[\chi^{\pm}]$ module the action of χ on the generators is multiplication by $x_{\rho_1}x_{\rho_2}$.

2.3. Surfaces. We consider the case of projective plane and the Hirzebruch surfaces (including $\mathbb{P}^1 \times \mathbb{P}^1$). The underlying torus T is two dimensional and in co-ordinates $R(T) = \mathbb{Z}[\chi_1^{\pm}, \chi_2^{\pm}]$. We identify $K_*(k) \otimes R(T)$ with $K_*(k)[\chi_1^{\pm}, \chi_2^{\pm}]$. We use $K_*(k)[\chi^{\pm}]$ to denote $K_*(k)[\chi_1^{\pm}, \chi_2^{\pm}]$, and with this notation, the right hand-side of eqn.(6) is

$$\prod_{\sigma_{ij} \in \Delta_{\max}} K_*(k) \otimes R(T_{\sigma_{ij}}) = K_*(k)[\chi^{\pm}]|_{\sigma_{12}} \times \dots \times K_*(k)[\chi^{\pm}]|_{\sigma_{ij}}$$

with a diagonal $K_*(k)[\chi^{\pm}]$ action. When no confusion is likely, we denote the ring $\prod_{\sigma_{ij} \in \Delta_{\max}} K_*(k) \otimes R(T_{\sigma_{ij}})$ will be denoted by $K_*(k)[\chi_{\sigma}^{\pm}]$.

2.3.1. The projective plane: \mathbb{P}^2 . The fan of \mathbb{P}^2 is shown in Figure 2.

GKM presentation. Explicitly the left-hand-side of eqn.(6) is given by

$$\begin{aligned} K_*(k) \otimes R(T_{\rho_1}) &= K_*(k)[\chi_1^{\pm}] \\ K_*(k) \otimes R(T_{\rho_2}) &= K_*(k)[\chi_2^{\pm}] \\ K_*(k) \otimes R(T_{\rho_3}) &= K_*(k)[(\chi_1\chi_2)^{\pm}]. \end{aligned}$$

An element $(f_1, f_2, f_3) \in K_*(k)[\chi_{\sigma}^{\pm}]$ belongs to $K_{T,*}(\mathbb{P}^2)$ if and only if $f_1 - f_2 = 0 \pmod{1 - \chi_1}$, $f_1 - f_3 = 0 \pmod{1 - \chi_2}$ and $f_2 - f_3 = 0 \pmod{1 - \chi_1^{-1}\chi_2}$.

RS presentation. The Reisner-Stanley presentation is given by

$$\frac{K_*(k)[x_{\rho_1}^{\pm}, x_{\rho_2}^{\pm}, x_{\rho_3}^{\pm}]}{(x_{\rho_1} - 1)(x_{\rho_2} - 1)(x_{\rho_3} - 1)}$$

where the generators x_{ρ_i} are mapped, via eqn.(9), to

$$\begin{aligned} x_{\rho_1} &\mapsto (\chi_1, 1, \chi_1 \chi_2^{-1}) \\ x_{\rho_2} &\mapsto (\chi_2, \chi_1^{-1} \chi_2, 1) \\ x_{\rho_3} &\mapsto (1, \chi_1^{-1}, \chi_2^{-1}) \end{aligned}$$

As a $K_*(k)[\chi^{\pm}]$ module, χ_1 action is multiplication by $x_{\rho_1} x_{\rho_3}^{-1}$ and χ_2 action is multiplication by $x_{\rho_2} x_{\rho_3}^{-1}$.

2.3.2. *The surface:* $\mathbb{P}^1 \times \mathbb{P}^1$. The fan of $\mathbb{P}^1 \times \mathbb{P}^1$ is shown in Figure 3.

GKM presentation. The left-hand-side of eqn.(6) is given by

$$\begin{aligned} K_*(k) \otimes R(T_{\rho_1}) &= K_*(k)[\chi_1^{\pm}] \\ K_*(k) \otimes R(T_{\rho_2}) &= K_*(k)[\chi_2^{\pm}] \\ K_*(k) \otimes R(T_{\rho_3}) &= K_*(k)[\chi_1^{\pm}] \\ K_*(k) \otimes R(T_{\rho_4}) &= K_*(k)[\chi_2^{\pm}]. \end{aligned}$$

The relations are given by: an element $(f_1, f_2, f_3, f_4) \in K_*(k)[\chi^{\pm}]$ belongs to $K_{T,*}(\mathbb{P}^1 \times \mathbb{P}^1)$ if and only if $f_1 - f_2 = 0 \pmod{(1 - \chi_1)}$, $f_2 - f_3 = 0 \pmod{(1 - \chi_2)}$, $f_3 - f_4 = 0 \pmod{(1 - \chi_1)}$ and $f_4 - f_1 = 0 \pmod{(1 - \chi_2)}$.

RS presentation. The Reisner-Stanley presentation is given by the formula

$$\frac{K_*(k)[x_{\rho_1}^{\pm}, x_{\rho_2}^{\pm}, x_{\rho_3}^{\pm}, x_{\rho_4}^{\pm}]}{((x_{\rho_1} - 1)(x_{\rho_3} - 1), (x_{\rho_2} - 1)(x_{\rho_4} - 1))}.$$

where the generators x_{ρ_i} are mapped as follows

$$\begin{aligned} x_{\rho_1} &\mapsto (\chi_1, 1, 1, \chi_1) \\ x_{\rho_2} &\mapsto (\chi_2, \chi_2, 1, 1) \\ x_{\rho_3} &\mapsto (1, \chi_1^{-1}, \chi_1^{-1}, 1) \\ x_{\rho_4} &\mapsto (1, 1, \chi_2^{-1}, \chi_2^{-1}). \end{aligned}$$

In this case, as a $K_*(k)[\chi^{\pm}]$ module, χ_1 action is multiplication by $x_{\rho_1} x_{\rho_3}^{-1}$ and χ_2 action is multiplication by $x_{\rho_2} x_{\rho_4}^{-1}$.

2.3.3. *The Hirzebruch surfaces:* \mathbb{F}_n , $n > 1$. The fan of a Hirzebruch surface \mathbb{F}_n is shown in Figure 4.

GKM presentation. Using the co-ordinates χ_1 and χ_2 on the torus the left-hand-side of eqn.(6) is given by

$$\begin{aligned} K_*(k) \otimes R(T_{\rho_1}) &= K_*(k)[\chi_1^\pm] \\ K_*(k) \otimes R(T_{\rho_2}) &= K_*(k)[\chi_2^\pm] \\ K_*(k) \otimes R(T_{\rho_3}) &= K_*(k)[(\chi_1^{-1}\chi_2^n)^\pm] \\ K_*(k) \otimes R(T_{\rho_4}) &= K_*(k)[\chi_2^\pm]. \end{aligned}$$

An element $(f_1, f_2, f_3, f_4) \in K_*(k)[\chi_\sigma^\pm]$ belongs to $K_{T,*}(\mathbb{F}_n)$ if and only if $f_1 - f_2 = 0 \pmod{(1 - \chi_1)}$, $f_2 - f_3 = 0 \pmod{(1 - \chi_1^n \chi_2)}$, $f_3 - f_4 = 0 \pmod{(1 - \chi_1)}$ and $f_4 - f_1 = 0 \pmod{(1 - \chi_2)}$.

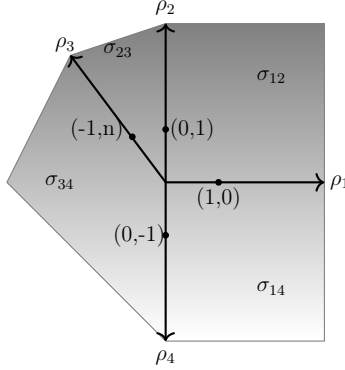


FIGURE 4. The fan of Hirzebruch surface \mathbb{F}_n .

RS presentation. The Reisner-Stanley presentation is given by

$$\frac{K_*(k)[x_{\rho_1}^\pm, x_{\rho_2}^\pm, x_{\rho_3}^\pm, x_{\rho_4}^\pm]}{((x_{\rho_1} - 1)(x_{\rho_3} - 1), (x_{\rho_2} - 1)(x_{\rho_4} - 1))}.$$

The generators x_{ρ_i} are mapped as follows

$$\begin{aligned} x_{\rho_1} &\mapsto (\chi_1, 1, 1, \chi_1) \\ x_{\rho_2} &\mapsto (\chi_2, \chi_1^n \chi_2, 1, 1) \\ x_{\rho_3} &\mapsto (1, \chi_1^{-1}, \chi_1^{-1}, 1) \\ x_{\rho_4} &\mapsto (1, 1, \chi_1^{-n} \chi_2^{-1}, \chi_2^{-1}). \end{aligned}$$

As a $K_*(k)[\chi^\pm]$ module, χ_1 action is multiplication of $x_{\rho_1} x_{\rho_3}^{-1}$ and χ_2 action is multiplication by $x_{\rho_2} x_{\rho_3}^n x_{\rho_4}^{-1}$.

3. TORUS ACTIONS ON SPHERICAL VARIETIES

In this section we will analyze the irreducible components of the *fixed point locus* X^S of diagonalizable subgroups S of a reductive group G and a G -variety X .

More precisely, for any G -variety X and any closed subgroup $S \subset G$ we consider the functor \underline{X}^S which associates to any affine k -scheme A the set

$$\underline{X}^S(A) = \{x \in X(A) \mid s \cdot x = x \text{ for any } s \in S(A)\}.$$

The functor \underline{X}^S is a representable closed sub-functor of \underline{X} . The *fixed point locus* of S on X is the closed subscheme X^S of X , representing \underline{X}^S , with the reduced scheme structure.³

We recall some standard results about reductive algebraic groups over fields. The main reference for these results is [Bor91].

Definition 3.1. Let S be any torus in G . Then S is called *regular* if S contains a regular element⁴ of G and S is called *singular* if S is contained in infinitely many Borel subgroups of G .

Remark 3.2. We note that Borel, in [Bor91, IV, §13.7, Cor. 2], considers a more general notion of a *semi-regular* torus but it turns out that for reductive groups a semi-regular torus is regular. So, for reductive groups, we have a dichotomy: a torus S is either regular or singular.

A maximal torus is always regular. So a singular torus S , contained in a maximal torus T , is a subtorus of codimension at-least one. All codimension one singular tori, contained in a fixed maximal torus T , correspond to the neutral component of $\text{Ker}(\alpha)$ where α is a root of G (with respect to T).

Centralizers. Suppose S is a codimension one subtorus of a maximal torus $T \subset G$. Then the centralizer $C_G(S)$ is a connected reductive group. If S is regular, then $C_G(S) = T$, and if S is singular, then the semisimple part $C_G(S)^{ss}$ has rank one. If B is any Borel subgroup of G containing S then $C_B(S)$ maps onto a Borel subgroup of $C_G(S)^{ss}$ and conversely any Borel subgroup of $C_G(S)^{ss}$, containing the image of S , is the image of a group of the form $C_B(S)$.

3.1. Spherical Varieties.

Definition/Proposition 3.3 (See [Bri86]). An irreducible G -variety X is called a *spherical variety* if any of the following equivalent conditions hold.

- (1) For any Borel subgroup B of G , the only B -invariant rational functions on X are constant functions.
- (2) The minimal codimension of a B -orbit in X is zero i.e. X has a dense open B -orbit.
- (3) X has finitely many B -orbits.

A *homogeneous spherical variety* is a homogeneous space G/H which is also a spherical G -variety.

The open G -orbit of a spherical variety X is a homogeneous spherical variety G/H and this is equivalent to the condition that the subvariety $B \cdot H$ is open in G . The subgroups $H \subset G$ satisfying this property are called *spherical subgroups*. It follows from Condition (3) and some additional arguments⁵ that the closure of a G -orbit inside a spherical variety is a spherical variety.

The set of B -eigenvalues of the eigenvectors in $k(X)$, the field of rational functions on X , is a sublattice of the lattice of characters of B . The rank of this lattice is a fundamental invariant of the spherical variety X and we will denote it by $r(X)$.

³It was pointed out by Brion that a result of Fogarty implies that X^S is already smooth; see [Fog73].

⁴Recall an element $g \in G$ is regular if the dimension of the centralizer $C_G(g)$ is minimal.

⁵It is not immediate that the closure is a normal variety.

3.1.1. *Torus action on spherical varieties.* Consider a spherical G -variety X . Throughout this section we fix a Borel subgroup B , a maximal torus $T \subset B$ and let S denote any codimension one subtorus of T .

Lemma 3.4. *The set of T -fixed points of a spherical variety X is finite.*

Proof. The spherical variety X decomposes as a finite union of G -orbits. So the Lemma is equivalent to showing that: if $x \in X$ is any torus fixed point then the G -orbit $G \cdot x$ has finitely many torus fixed points.

The latter statement follows from Lemma 2.2 of [DCS85], which shows that a G -homogeneous space $G \cdot x$ has finitely many T fixed points if the stabilizer of x contains T . \square

Remark 3.5. If X is complete then Borel Fixed Point Theorem shows that X^T is nonempty. In particular, X^T has at least $\dim(X) + 1$ points, [Bor91, Theorem 10.2, IV].

Now consider the fixed point locus X^S . It is stable under the action of the centralizer $C_G(S)$ and the structure of X^S , somewhat unsurprisingly, depends on whether S is regular or singular.

Lemma 3.6. *Let x be a S -fixed point of X . The intersection of $B \cdot x$ with the fixed point loci X^S is the $C_B(S)$ orbit $C_B(S) \cdot x$.*

Proof. It is clear that $C_B(S) \cdot x \subseteq B \cdot x \cap X^S$.

Let us fix a realization, à la Springer [Spr09, Chapter 8], of G with respect to the root system $\Phi := \Phi(G, T)$ and Φ^+ is the subset of positive roots. This gives us a family of homomorphisms $\varphi_\alpha : \mathbb{G}_a \rightarrow G$, indexed by $\alpha \in \Phi$, such that for any $t \in T$ we get $t \cdot \varphi_\alpha \cdot t^{-1} = u_\alpha(\alpha(t) \cdot x)$. The Borel subgroup B admits a decomposition $B = TU$ and U is generated by the images of $(\varphi_\alpha)_{\alpha \in \Phi^+}$. To prove the converse assertion let $y \in B \cdot x \cap X^S$. Then $y = b_0 \cdot x$ for some $b_0 \in B$. We write $b_0 = t_0 \cdot u_0$ where $t_0 \in T$, $u_0 \in U$ and we can moreover assume $u_0 = \varphi_\alpha(z_0)$ for some $z_0 \in \mathbb{G}_a$.

We have

$$s \cdot b_0 \cdot s^{-1} = t_0 \cdot \varphi_\alpha(\alpha(s)z_0).$$

When S is a singular torus and α is a root such that $S \subset \text{Ker}(\alpha)$ then clearly $b_0 \in C_B(S)$. When α is a nontrivial character of S we have, for any $s \in S$,

$$y = s \cdot y = t_0 \varphi_\alpha(\alpha(s)z_0) \cdot x = \lim_{s \rightarrow 0} t_0 \varphi_\alpha(\alpha(s)z_0) \cdot x = t_0 \cdot x.$$

This shows that we can write $y = t_0 \cdot x$ for some $t_0 \in T$ and the conclusion follows. \square

Corollary 3.7. *We continue to use the notation of Lemma 3.6. In this case, if Y is any irreducible component of X^S then Y is a spherical $C_G(S)$ -variety.*

Proof. The normality of Y follows from [Fog73]. The group $C_G(S)$ is connected, so it stabilizes Y . The ambient variety X is spherical hence it has finitely many B -orbits; therefore using the previous lemma we see that Y has finitely many $C_B(S)$ orbits. Hence Y is a spherical $C_G(S)$ variety. \square

Corollary 3.8. *We continue with the notation of Corollary 3.7. The variety Y has a dimension at most two and as a $C_G(S)$ -spherical variety it has rank $r(Y)$ at most one.*

Proof. The dimension of Y is bounded by the dimension of $C_G(S)$. The assertion is clear when the quotient $C_G(S)/S$ has dimension one. Consider the case when S is singular and hence $\dim(C_G(S)/S)$ is greater than one. In this case the dimension and rank of Y are determined by the dimension and rank of the open $C_G(S)/S$ -orbit.

The center Z_S of $C_G(S)$ is contained in S . As a result, the action of $C_G(S)$ on Y factors through the semisimple part $\mathcal{G}_S := C_G(S)^{ss}$. The group $C_G(S)^{ss}$ has semisimple rank one and hence $\mathcal{G}_S = SL_2$ or PSL_2 . Let \mathcal{B}_S denote the image of $C_B(S)$ in \mathcal{B}_S .

The dimension of Y is bounded by the dimension of \mathcal{B}_S which is two, and the rank of Y is bounded by the dimension of the character lattice of \mathcal{B}_S which is at most one. This proves the assertion. \square

3.2. Rank one spherical varieties. A complete classification of rank one spherical varieties is known, see [Ahi83, Bri87] for details. In this section, we drop the sub-script S from the semi-simple group \mathcal{G}_S , the stabilizer \mathcal{H}_S etc. When \mathcal{G} is SL_2 or PSL_2 ⁶ the classification is determined by the spherical subgroups of \mathcal{G} . These rank one compactifications will be used in the next section. We recall the spherical subgroups, equivariant compactifications and the boundary of the open \mathcal{G} orbit in Table 1.

Spherical subgroup $\mathcal{H} \subset \mathcal{G}$	Equivariant of \mathcal{G}/\mathcal{H}	Boundary of open \mathcal{G} orbit
$B =$ Borel subgroup	\mathbb{P}^1	
$T =$ Maximal Torus	$\mathbb{P}^1 \times \mathbb{P}^1$	diagonal \mathbb{P}^1
$N_G(T) =$ Normalizer of T	$\mathbb{P}(\mathfrak{sl}_2)$	conic of nilpotent matrices
$C_n \ltimes U$, where $C_n = \text{diag}(\zeta, \zeta^{-1})$ for $\zeta^n = 1$ and U is the unipotent subgroup	$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus 0)$ and $\mathbb{P}(0 \oplus \mathcal{O}_{\mathbb{P}^1}(n))$

TABLE 1. Equivariant embeddings of semisimple rank one groups.

Let us continue with the notation of the proof of Corollary 3.8. Let y_0 be a generic point in the open \mathcal{G} orbit and consider the orbit map $\varphi : \mathcal{G}/\mathcal{H} \rightarrow Y$ given by $g \mapsto g \cdot y$. Let \mathcal{X} denote one of the equivariant compactifications in the Table 1 (depending on \mathcal{H}). We consider the birational \mathcal{G} -equivariant map $\varphi : \mathcal{X} \dashrightarrow Y$. The map φ is possibly undefined in a codimension two locus by Zariski's Main Theorem. When \mathcal{X} is two-dimensional the boundary is one dimensional so the birational map φ extends by \mathcal{G} -equivariance to the boundary.

When \mathcal{X} is not the surface $\mathbb{P}^1 \times \mathbb{P}^1$ the boundary curves are not contractible (-1) curves and \mathcal{X} is a minimal model. So the map φ is necessarily an isomorphism. In the case $\mathcal{X} = \mathbb{P}^1 \times \mathbb{P}^1$ any contractible curve necessarily intersects the dense \mathcal{G} orbit. So the map φ is an isomorphism.

We get a complete characterization of the geometry of the irreducible two dimensional components of X^S . Summarizing, we have the following proposition.

⁶We use \mathcal{G} to denote the semi-simple rank one group case. The is to distinguish from the arbitrary reductive group case, denoted by G , in the next section.

Proposition 3.9. *Suppose $S \subset T$ is a codimension one subtorus, and let $Y \subseteq X^S$ be an irreducible. Then up-to isomorphism Y is one of the following varieties.*

- (i) *a point,*
- (ii) *a smooth \mathbb{P}^1 , identified with the complete flag variety \mathcal{G}/\mathcal{B} .*
- (iii) *a projective plane on which $C_G(S)$ acts through the projectivization of the adjoint action on the Lie-algebra of SL_2 ,*
- (iv) *a Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ for $n > 1$ (or $\mathbb{P}^1 \times \mathbb{P}^1$) identified with the projectivization of the rank two equivariant vector bundle $\mathcal{G} \times_{\mathcal{B}} V$ over $\mathcal{G}/\mathcal{B} \rightarrow B$. Here $V = k \cdot \chi \oplus k \cdot \chi^n$, for $n \geq 1$, is a two-dimensional B -representation (extended trivially from T) and χ is a generator of the lattice of characters of T .*

4. K-THEORY OF SPHERICAL VARIETIES

In this section our goal is to prove Theorem 1.1. This will be achieved in two steps. In the first step, we will provide a direct proof for the two-dimensional rank-one spherical varieties which appear in Proposition 3.9. Then using these results, combined with the work in previous sections, we will prove the general case.

4.1. Rank one case. Let us first outline our strategy. Suppose \mathcal{X} denote any of the two dimensional compactifications listed in Proposition 3.9 then it follows that \mathcal{X} is also a toric variety compactifying a two-dimensional torus T . Let \mathcal{T} denote a fixed maximal torus of \mathcal{G} and \mathcal{T} -acts on \mathcal{X} via the \mathcal{G} -action.

We will show that there is a closed embedding $\iota : \mathcal{T} \hookrightarrow T$ which makes the Diagram (12), where the vertical maps are the action maps, commutative.

$$(12) \quad \begin{array}{ccc} \mathcal{X} \times \mathcal{T} & \xrightarrow{id \times i} & \mathcal{X} \times T \\ \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{id} & \mathcal{X} \end{array}$$

When we have such a map ι , Proposition 2.5 shows that

$$(13) \quad K_{T,*}(\mathcal{X}) \otimes_{R(T)} R(\mathcal{T}) \simeq K_{\mathcal{T},*}(\mathcal{X}).$$

The structure of $K_{T,*}(\mathcal{X})$ is explicit from the toric computations in Section 2.1 and the $R(T)$ -module structure on $R(\mathcal{T})$ is clear (it depends on ι).

We fix co-ordinates which identifies \mathcal{T} with the maximal torus

$$T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{G}_m \right\}.$$

in \mathcal{G} and let χ denote the character

$$\chi \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = t$$

which is a generator of the character lattice of \mathcal{T} and hence $R(\mathcal{T}) = \mathbb{Z}[\chi^{\pm}]$. Recall that, using co-ordinates χ_1, χ_2 , we have $R(T) = \mathbb{Z}[\chi_1^{\pm}, \chi_2^{\pm}]$.

The key to constructing the embedding $\iota : \mathcal{T} \hookrightarrow T$ is the fact that for all varieties, listed in Proposition 3.9, the complement of the open \mathcal{G} -orbit is a \mathcal{G} -stable hence \mathcal{T} -stable hypersurface. This imposes enough restrictions on ι to recover the map.

4.1.1. *The case of $\mathcal{X} = \mathbb{P}(\mathfrak{sl}_2)$.* We identify the coordinates $[x_0 : x_1 : x_2]$ on $\mathbb{P}(\mathfrak{sl}_2)$ with trace zero matrices: $\begin{pmatrix} x_0 & x_1 \\ x_2 & -x_0 \end{pmatrix}$ modulo scalars. The torus T acts on \mathbb{P}^2 is by $t \cdot [x_0 : x_1 : x_2] = [x_0 : \chi_1(t) \cdot x_1 : \chi_2(t) \cdot x_2]$ and the torus \mathcal{T} acts on \mathfrak{sl}_2 by conjugation action. The complement of the open \mathcal{G} orbit is isomorphic to the quadric $x_0^2 + x_1x_2 = 0$. Hence the embedding $\iota : \mathcal{T} \hookrightarrow T$ is given by $t \mapsto (t, t^{-1})$. The induced map on the representation rings is given by

$$\begin{aligned} \mathbb{Z}[\chi_1^\pm, \chi_2^\pm] &\rightarrow \mathbb{Z}[\chi^\pm] \\ \chi_1 &\mapsto \chi \\ \chi_2 &\mapsto \chi^{-1}. \end{aligned}$$

The GKM presentation follows immediately from Proposition 2.5.

Proposition 4.1. *The \mathcal{T} equivariant K -theory of \mathbb{P}^2 is given by 3-tuples (f_1, f_2, f_3) where each f_i (for $i = 1, 2, 3$) belongs to the ring $K_*(k)[\chi^\pm]$, which satisfies the relations*

$$f_1 - f_2 = 0 \pmod{(1 - \chi)}, \quad f_1 - f_3 = 0 \pmod{(1 - \chi)}, \quad \text{and} \quad f_2 - f_3 = 0 \pmod{(1 - \chi^2)}.$$

4.1.2. *The case of $\mathcal{X} = \mathbb{P}^1 \times \mathbb{P}^1$.* The diagonal \mathbb{P}^1 in \mathcal{X} is the boundary of the open \mathcal{G} -orbit. The diagonal intersects the dense T orbit. As a result, the only map that preserves the boundary of the compactification is the diagonal map. The induced map on the representation rings is given by

$$\begin{aligned} \mathbb{Z}[\chi_1^\pm, \chi_2^\pm] &\rightarrow \mathbb{Z}[\chi^\pm] \\ \chi_1 &\mapsto \chi \\ \chi_2 &\mapsto \chi. \end{aligned}$$

The GKM presentation follows immediately from Proposition 2.5.

Proposition 4.2. *The \mathcal{T} equivariant K -theory of $\mathbb{P}^1 \times \mathbb{P}^1$ is given by a 4-tuple (f_1, f_2, f_3, f_4) , where each f_i (for $i = 1, 2, 3, 4$) belongs to the ring $K_*(k)[\chi^\pm]$. Moreover all such tuples satisfy the relation*

$$f_i - f_j = 0 \pmod{(1 - \chi)} \text{ for } i, j \in \{1, 2, 3, 4\}.$$

4.1.3. *The case of $\mathcal{X} = \mathbb{F}_n$.* Let $\mathcal{B}^- \subset \mathcal{G}$ denote the Borel subgroup of lower triangular matrices. Consider the character ϕ_n of \mathcal{B}^- defined by

$$\phi_n \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} = a^n;$$

so $\phi_n = \chi^n$. The group \mathcal{B}^- acts on \mathbb{P}^1 by $b \cdot [x_0 : x_1] = [x_0 : \phi_n(b)x_1]$ and the associated bundle $\mathcal{G} \times_{\mathcal{B}^-} \mathbb{P}^1$, with the canonical fibration to $\mathcal{G}/\mathcal{B}^- = \mathbb{P}^1$ is identified with the Hirzebruch surface \mathbb{F}_n .

The map $\iota : \mathcal{T} \rightarrow T$ is given by $t \mapsto (t, t^n)$.

The induced map on the representation rings is given by

$$\begin{aligned} \mathbb{Z}[\chi_1^\pm, \chi_2^\pm] &\rightarrow \mathbb{Z}[\chi^\pm] \\ \chi_1 &\mapsto \chi \\ \chi_2 &\mapsto \chi^n. \end{aligned}$$

The following proposition is immediate from Proposition 2.5.

Proposition 4.3. *The \mathcal{T} equivariant K -theory of \mathbb{F}_n is given by a 4-tuple of elements in (f_1, f_2, f_3, f_4) where each f_i (for $i = 1, 2, 3, 4$) belongs to the ring $K_*(k)[\chi^\pm]$, and satisfies the relations*

$$f_1 - f_2 = 0 \pmod{(1 - \chi)}, \quad f_2 - f_3 = 0 \pmod{(1 - \chi^{2n})}, \quad f_3 - f_4 = 0 \pmod{(1 - \chi)},$$

$$\text{and } f_4 - f_1 = 0 \pmod{(1 - \chi^n)}.$$

The associated RS presentations are also easily computed.

Proposition 4.4 (Reisner-Stanley presentation). *The torus equivariant K -theory of wonderful rank one \mathcal{G} compactifications have the following presentations.*

$$K_{\mathcal{T},*}(\mathbb{P}^2) = \frac{K_*(k)[x_{\rho_1}^\pm, x_{\rho_2}^\pm, x_{\rho_3}^\pm]}{((x_{\rho_1}x_{\rho_2}x_{\rho_3}^{-2} - 1), (x_{\rho_1} - 1)(x_{\rho_2} - 1)(x_{\rho_3} - 1))}$$

$$K_{\mathcal{T},*}(\mathbb{P}^1 \times \mathbb{P}^1) = \frac{K_*(k)[x_{\rho_1}^\pm, x_{\rho_2}^\pm, x_{\rho_3}^\pm, x_{\rho_4}^\pm]}{((x_{\rho_1}x_{\rho_3}^{-1} - x_{\rho_2}x_{\rho_4}^{-1}), (x_{\rho_1} - 1)(x_{\rho_3} - 1), (x_{\rho_2} - 1)(x_{\rho_4} - 1))}$$

$$K_{\mathcal{T},*}(\mathbb{F}_n) = \frac{K_*(k)[x_{\rho_1}^\pm, x_{\rho_2}^\pm, x_{\rho_3}^\pm, x_{\rho_4}^\pm]}{((x_{\rho_1}^n x_{\rho_3}^{-n} - x_{\rho_2} x_{\rho_3}^n x_{\rho_4}^{-1}), (x_{\rho_1} - 1)(x_{\rho_3} - 1), (x_{\rho_2} - 1)(x_{\rho_4} - 1))}.$$

Proof. The proposition follows from the fact that we have a presentation $R(\mathcal{T}) = R(T)/I$. As a result, by using Proposition 2.5, we get a presentation of the equivariant groups $K_{\mathcal{T},*}(\mathcal{X}) = K_{T,*}(\mathcal{X})/I$. The explicit description of the ideal I then follows from the explicit calculations of the $R(T)$ -module structure on the Reisner-Stanley presentation carried out in Sections 2.3.1 - 2.3.3.

Let us work out the case of \mathbb{P}^2 . The ideal is generated by $(1 - \chi_1\chi_2)$ and using the $R(T)$ -module structure of the RS presentation, worked out in Section 4.1.1, we get

$$R(\mathcal{T}) = \frac{\mathbb{Z}[\chi_1^\pm, \chi_2^\pm]}{(1 - \chi_1\chi_2)}.$$

The other cases are similar and this proves the proposition. \square

4.2. Weyl group action. The Weyl group of \mathcal{G} acts on \mathcal{X} . Let w_0 denote the non trivial element in the Weyl group. It permutes the torus fixed points and consequently acts on the ring $K_*(k)[\chi_\sigma^\pm]$ (recall the notation used in Section 2.3). So, to understand the action of the Weyl group on the \mathcal{T} -equivariant K -theory we need to understand the action on the torus fixed points.

Proposition 4.5. *Let w_0 be the non-trivial element of the Weyl group of \mathcal{G} . Then the w_0 action on the compactifications \mathcal{X} are given as follows. Let σ_{ij} denote the maximal cones of the fans in Figures 2, 3 and 4 and $x_{\sigma_{ij}}$ denote the corresponding torus fixed points of the toric varieties.*

- (1) If $\mathcal{X} = \mathbb{P}^1$ then w_0 permutes the two torus fixed point.
- (2) If $\mathcal{X} = \mathbb{P}^2$ then $w_0(x_{\sigma_{12}}) = x_{\sigma_{12}}$ and w permutes the other two torus fixed points i.e. $x_{\sigma_{23}} \xleftrightarrow{w_0} x_{\sigma_{13}}$.
- (3) If $\mathcal{X} = \mathbb{P}^1 \times \mathbb{P}^1$ then $x_{\sigma_{12}} \xleftrightarrow{w_0} x_{\sigma_{34}}$ and it leaves the other two torus fixed points $x_{\sigma_{23}}$ and $x_{\sigma_{14}}$ invariant.

- (4) If $\mathcal{X} = \mathbb{F}_n$ then $x_{\sigma_{12}} \xleftrightarrow{w_0} x_{\sigma_{34}}$ and it leaves the other two torus fixed points $x_{\sigma_{23}}$ and $x_{\sigma_{14}}$ invariant.

Proof. The main ingredient in the proof is a “dynamical” interpretation of the torus fixed points in a toric variety. Let us denote this torus fixed point, associated to a maximal cone σ in the fan Δ , by x_σ . Let $e \in X(\Delta)$ be any point in the dense torus T of $X(\Delta)$. The point x_σ is also the limit of the one parameter orbit $\nu(t) \cdot e$ where ν is any co-character in the interior of the cone σ .

The embedding $\iota : \mathcal{T} \rightarrow T$, considered in the diagram (12), defines a co-character $i_\chi := \iota(\chi)$ in N_T . We have a decomposition of lattices $N_T = \mathbb{Z} \cdot i_\chi \oplus \mathbb{Z} \cdot i_\chi^\perp$. We extend the action of w_0 to N_T by defining $w_0(i) = -i$ and $w_0(i^\perp) = i^\perp$. Any co-character of $\lambda \in N_{T^b}$ decomposes uniquely as $n \cdot i + m \cdot i^\perp$. Hence we can calculate the w_0 action by

$$w_0(x_\sigma) = \lim_{t \rightarrow 0} w_0(\lambda)(t)$$

where λ is any co-character in the interior of σ .

The proposition follows by easy computations in a case-by-case analysis. \square

This allows us to completely calculate the \mathcal{G} -equivariant K -theory of rank one \mathcal{G} wonderful varieties.

Proposition 4.6. *Let \mathcal{X} be a rank one wonderful compactification of \mathcal{G} . Then its \mathcal{G} equivariant K -theory is given as follows.*

- (1) If $\mathcal{X} = \mathbb{P}^1$ then $K_{\mathcal{G},*}(\mathcal{X}) = K_*(k) \otimes R(T)$.
- (2) If $\mathcal{X} = \mathbb{P}^2$, the $K_{\mathcal{G},*}(\mathcal{X})$ is given by a collection of elements in (f_1, f_2) in the ring $\prod_{i=1}^2 K_*(k)[\chi^\pm]$ which satisfies the condition

$$f_1 - f_2 = 0 \pmod{(1 - \chi)}.$$

- (3) If $\mathcal{X} = \mathbb{P}^1 \times \mathbb{P}^1$, then $K_{\mathcal{G},*}(\mathcal{X})$ is given by a collection of elements (f_1, f_2, f_3) in the ring $\prod_{i=1}^3 K_*(k)[\chi^\pm]$ which satisfies the condition

$$f_i - f_j = 0 \pmod{(1 - \chi)} \text{ for all } i, j \in \{1, 2, 3\}.$$

- (4) If $\mathcal{X} = \mathbb{F}_n$, then $K_{\mathcal{G},*}(\mathcal{X})$ is given by a collection of elements in (f_1, f_2, f_3) in the ring $\prod_{i=1}^3 K_*(k)[\chi^\pm]$ which satisfies the conditions

$$f_1 - f_2 = 0 \pmod{(1 - \chi^{2n})}, \text{ and } f_3 - f_1 = 0 \pmod{(1 - \chi^n)}.$$

Proof. The first assertion is well known and see Example 4.12 below for an outline of the proof. The second and the third part of this proposition are direct consequences of Proposition 4.5, Proposition 4.2 and Proposition 4.1.

We consider the final part. Let (f_1, f_2, f_3, f_4) be an element in $\prod_{i=1}^4 K_*(k)[\chi^\pm]$ which satisfies the conditions of Proposition 4.5. The invariants, described in Proposition 4.5, impose additional relations $f_1 = f_3$ and hence $f_3 - f_2 = f_1 - f_2 = 0 \pmod{(1 - \chi^{2n})}$. This proves the proposition. \square

Remark 4.7. It is tedious, but not difficult to work out the explicit Weyl group action on the Reisner-Stanley presentations. However the variables x_ρ are not very well adapted to the this action. We will not use this action so we omit the details.

4.3. General case. Let us now return to the general case of a reductive group G and a fixed maximal torus $T \subset G$ and a G -variety X . In this case the T -equivariant K -theory admits a concrete description. The G -equivariant K -theory is much more subtle. The problem is that X admits a stratification by several G -orbits and only some of them contain T -fixed points. In general the stabilizer of the Weyl-group action on the torus fixed points in each strata will be different. To the best of our knowledge there is no uniform way to handle this issue; however see [Str92, BBJ16] for the case of complete quadrics and Section 5 for minimal rank symmetric varieties. We have the following result.

Theorem 4.8. *Let X be a smooth projective spherical G -variety. Let T be a maximal torus of G . Then the set of T fixed points X^T is finite and the T equivariant K -theory $K_{T,*}(X)$ is an ordered set of elements $(f_x)_{x \in X^T}$ from $\prod_{x \in X^T} K_*(k) \otimes R(T)_x$ which are subject to the following additional congruences:*

- $f_x - f_y = 0 \pmod{(1 - \chi)}$ when x and y are connected by a T -invariant curve of weight χ .
- $f_x - f_y = f_x - f_z = 0 \pmod{(1 - \chi)}$ and $f_y - f_z = 0 \pmod{(1 - \chi^2)}$ where χ is a root of the pair (G, T) . The irreducible component of the subvariety $X^{\text{Ker}(\chi)}$ which contains the points x, y, z is isomorphic to \mathbb{P}^2 and there is an element in the Weyl group of G that fixes x and permutes the point y and z .
- $f_x - f_y = f_y - f_z = f_z - f_w = f_x - f_w = 0 \pmod{(1 - \chi)}$ where χ is a root of the pair (G, T) . The irreducible component of the subvariety $X^{\text{Ker}(\chi)}$ which contains the points x, y, z, w is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and there is an element in the Weyl group of G that fixes two points and permutes the other two.
- $f_x - f_y = f_z - f_w = 0 \pmod{(1 - \chi)}$ and $f_y - f_z = 0 \pmod{(1 - \chi^{2n})}$ and $f_x - f_w = 0 \pmod{(1 - \chi^n)}$, where $n \geq 1$, and χ is a root of the pair (G, T) . The irreducible component of the subvariety $X^{\text{Ker}(\chi)}$ which contains the points x, y, z, w is isomorphic to a ruled surface \mathbb{F}_n . There is an element in the Weyl group of G that fixes the points x and w and permutes z and y .

The G -equivariant K -theory is given by the space of W -invariants in the T -equivariant K -groups.

Proof. Let S denote any codimension one subtorus of T . We know that the irreducible components of S fixed points X^S are either smooth curves \mathbb{P}^1 or one of the compactifications listed in Proposition 3.9.

The formula, from Proposition 2.7,

$$K_{T,*}(X^S) = K_{T/S,*}(X^S) \otimes_{R(T/S)} R(T)$$

reduces the problem to calculating $K_{T/S,*}(X^S)$ which follows from Section 4.1.

Note that in all these cases, the Weyl group of the neutral component of $C_G(S)$ (trivial when $C_G(S) = T$) embeds as a subgroup of W . The proposition then follows from the Proposition 4.5. \square

Remark 4.9. In the group compactification case, the connected components of X^S are only smooth curves. In this case the structure of the T -equivariant (and G -equivariant) K -theory has been worked out by Uma in [Uma07].

Reisner-Stanley Presentation. Informally speaking, the Reisner-Stanley presentation is “generated” by patching the RS presentation of each irreducible component of X^S as S varies over the codimension one subtori of T . To formalize this, we construct a topological space $PL(X)$ associated to X . It is a two-dimensional topological space. We start a collection of points corresponding to the T -fixed points of X . Then we glue the unit interval to pairs of points $\{x_\alpha, x_\beta\} \in X^T$ if there is a T -stable curve passing through them. Next we consider subsets of three or four points in X^T . If such a collection of points belong to an irreducible invariant surface then we glue a polyhedron, corresponding to the moment polytope of the corresponding toric surface, with the vertices of the polyhedron glued to the torus fixed point. The boundary of a two cell doesn’t necessarily glue to one cells so the space $PL(X)$ is not a simplicial complex. The Weyl group of G acts on $PL(X)$.

Proposition 4.10 (Reisner-Stanley presentation). *Let X be a smooth projective spherical G variety and let $PL(X)$ be the topological space constructed in the preceding paragraph. Let s be an one or two dimensional cell of $PL(X)$ and we let X_s denote the corresponding component. Then we have a map of rings*

$$\varphi_s : K_{T/S,*}(X_s) \otimes_{R(T/S)} R(T) \rightarrow K_{T,*}(X^T).$$

The image of the colimit of the family of the maps φ_s is identified with $K_{T,}(X)$.*

Proof. The T/S invariant points of X_s form a subset of X^T . Let us denote this subset by $X_s^{T/S}$. The map φ_s is the composition of maps, see Diagram (14), where the horizontal arrow is a result of Theorem 2.9 and the vertical arrow is inclusion.

$$(14) \quad \begin{array}{ccc} K_{T/S,*}(X_s) \otimes_{R(T/S)} R(T) & \longrightarrow & K_{T,*}(X_s^{T/S}) \otimes_{R(T/S)} R(T) \\ & \searrow \varphi_s & \downarrow \\ & & K_{T,*}(X^T) \end{array}$$

The proposition now follows from the fact that an element $(f_i) \in K_{T,*}(X^T)$ is in the image of $K_{T,*}(X)$ when it satisfies the relations imposed by Theorem 4.8. These relations only depend on the irreducible component containing a given collection of points and the image of the map φ_s , from the corresponding component, is a isomorphism (see Proposition 4.4). \square

Remark 4.11. The previous proposition gives an immediate set of generators for $K_{T,*}(X)$. However in general the W group invariants are harder to extract because the variables used in this presentation are not well adapted to this group action.

Example 4.12. We consider the complete flag variety to illustrate the various presentations of the equivariant K -theory. We fix a connected reductive group G , a Borel subgroup B and a maximal torus $T \subset B$. This fixes a root system Δ_G of G and we denote the associated Weyl group by W . Given any simple root $\alpha \in \Delta_G$ we denote the associated element in W by s_α . We let \mathcal{B} denote the complete flag variety G/B .

We note that, using Proposition 2.2 and the discussion after that we get $K_{G,*}(\mathcal{B}) = K_{B,*}(pt)$ and $K_{B,*}(pt) = K_{T,*}(pt) = K_*(k) \otimes R(T)$.

GKM presentation. The T -fixed points and T -stable curves were described by Carrell in [Car94, Theorem F]. It turns out, this data is given by the Bruhat graph W_G ; the vertices are the T -fixed points of \mathcal{B} hence they are indexed by elements of the Weyl group. Two

vertices w and w' are joined by a T -stable curve if $w' = s_\alpha \cdot w$ for some simple reflection s_α and $\ell(w) < \ell(w')$.

As a consequence of Theorem 4.8 we get

$$K_{T,*}(\mathcal{B}) = \left\{ (f_w) \in \prod_{w \in W} K_*(k) \otimes R(T) : \begin{array}{l} \text{where } f_w - f_{w'} = 0 \pmod{1 - \alpha} \text{ whenever} \\ w' = s_\alpha \cdot w \text{ and } \ell(w) < \ell(w'). \end{array} \right\}.$$

The Weyl group W acts transitively on $\prod_{w \in W} K_*(k) \otimes R(T)$ by permuting the co-ordinates. So it suffices to check the congruences imposed at the identity. Any element in $K_{T,*}(\mathcal{B})^W$ is determined by choosing $f_e \in R(T)$. So, $K_{T,*}(\mathcal{B})^W = K_*(k) \otimes R(T)$.

RS presentation. The topological space $PL(\mathcal{B})$ is just the topological realization of the Bruhat graph. Let s_α be an edge of the Bruhat graph joining vertices w_0 and $w_1 = w_0 \cdot s_\alpha$ (see Figure 5 below). The component X_{s_α} is isomorphic to \mathbb{P}^1 . We have RS presentation, following Section 2.2,

$$K_{T/S,*}(X_{s_\alpha}) = \frac{K_*(k)[x_{\alpha_1}^\pm, x_{\alpha_2}^\pm]}{(x_{\alpha_1} - 1)(x_{\alpha_2} - 1)} = K_*(k)$$

The map φ_{s_α} in this case is just the identity on the corresponding end-points and the T -equivariant cohomology is identified with

$$\prod_{e \in PL(\mathcal{B})} K_*(k) \otimes R(T)$$

where the indexing set is the set of edges of the Bruhat graph.

To compute the G -equivariant K -theory it suffices to look at the intersection of image of φ and the sub-ring $K_*(k) \otimes R(T)$ of $K_{T,*}(X^T)$ corresponding to the identity $e \in W$. The Bruhat graph near e is depicted in Figure 6. Since all the edges emanate from e we note that G -equivariant K -theory is determined by the factor $K_*(k) \otimes R(T)$ at the identity.

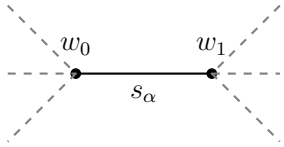


FIGURE 5. An edge of Bruhat graph connecting w_0 and $w_1 = w_0 \cdot s_\alpha$

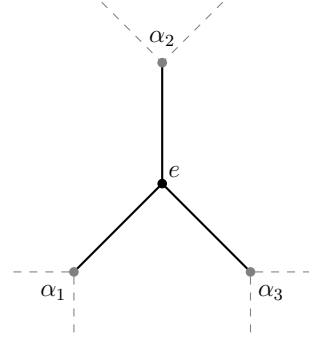


FIGURE 6. The local picture of Bruhat graph near the identity e . The vertices α_i are simple roots.

Remark 4.13. A consequence of Merkurjev's comparison result on ordinary and equivariant K -theory of algebraic varieties shows that smooth and projective varieties are *equivariantly*

formal⁷; see [Mer97, Proposition 3.6 and Theorem 4.2]. In particular, we can recover the ordinary K -theory of X from its T -equivariant K -theory by the formula

$$K_*(X) = K_{T,*}(X)/\mathfrak{m} \cdot K_{T,*}(X)$$

where \mathfrak{m} denotes the augmentation ideal of $R(T)$.

Alternatively, this also follows from Theorem 1.2 of [JK15].

5. APPLICATION: WONDERFUL COMPACTIFICATION OF SYMMETRIC SPACES OF MINIMAL RANK

In this section, we will apply our results to wonderful compactifications of symmetric spaces of minimal rank. We first recall some known facts about these varieties and refer the reader to the articles [DCS99, Spr, Ric82] for detailed proofs.

5.1. Symmetric spaces and their compactifications. Throughout this section G is a connected linear semisimple group of adjoint type and $\theta : G \rightarrow G$ be a non-trivial involution. Let $H = G^\theta$ denote the subgroup of θ -fixed points. The homogeneous space G/H is called a symmetric space. The group H is reductive and without loss of generality we assume that it is connected⁸.

It turns out that the homogeneous space G/H is a spherical G -variety and rank of G/H , as a spherical variety, is always constrained by the inequality

$$\text{rank}(G) \geq \text{rank}(H) + \text{rank}(G/H).$$

A *minimal rank* variety is a G -variety such that the above inequality is an equality. The rank doesn't change when one passes to a spherical equivariant compactification of G/H . Throughout this section we will consider wonderful compactifications of minimal rank symmetric spaces.

Definition 5.1. A θ -split torus of G is a θ -stable torus $S \subset G$ such that $\theta(s) = s^{-1}$. A θ -split parabolic P of G is a parabolic subgroup such that $P \cap \theta(P)$ = Levi subgroup of P (and $\theta(P)$).

It turns out that for reductive groups nontrivial θ -split torus always exist. Let us fix a maximal θ -split torus S and a minimal θ -split parabolic P containing S . Let L and P_u denote the Levi component, and the unipotent radical of P respectively. It follows from construction that $L = C_G(S)$. The derived group of L , denoted by $[L, L]$ is θ -stable and it has no θ -split torus. As a result, we conclude that $[L, L] \subset H$. We fix a maximal torus T of P containing S . As a consequence of the minimal rank assumption we see that $S \times (T \cap H) \rightarrow T$ is an isogeny.

Let Φ_G and Φ_L denote the root systems associated to the pairs (G, T) and (L, T) . The positive roots, the simple roots, and the associated Weyl groups are denoted by Φ_G^+ (resp. Φ_L^+), Δ_G (resp. Δ_L) and W_G (resp. W_L) respectively. The involution θ acts on roots and we have a subset

$$\Phi^{-\theta} := \{\alpha \in \Phi_G^+ \mid \theta(\alpha) < 0\}.$$

Since $[L, L] \subset H$ the action of θ is trivial on Φ_L and W_L . It turns out that we have a partition of positive roots

$$\Phi_G^+ = \Phi^{-\theta} \cup \Phi_L^+$$

⁷A notion first defined in [GKM98].

⁸Otherwise we replace H by its connected component.

and a compatible partition of simple roots

$$(15) \quad \Delta_G = \Delta_L \cup \Delta^{-\theta},$$

where $\Delta^{-\theta} := \Delta \cap \Phi^{-\theta}$.

The inclusion map $i : S \hookrightarrow T$ and the surjective homomorphism $p : T \rightarrow S$ is defined by $p(t) = t \cdot \theta(t)^{-1}$ fits into a commutative diagram

$$(16) \quad \begin{array}{ccccc} & & *^2 & & \\ & \nearrow & & \searrow & \\ S & \xrightarrow{i} & T & \xrightarrow{p} & S \end{array}$$

where $*^2$ is the squaring map.

The injective map $p : M_S \rightarrow M_T$ on characters identifies M_S with the subspace generated by the vectors $\alpha - \theta(\alpha)$. The nonzero vectors $i(\Phi_G)$ in the image of the surjective map $i : M_T \rightarrow M_S$ generate a (possibly non-reduced) root system, denoted by $\Phi_{G/H}$, and the image $\Delta_{G/H} := i(\Delta^{-\theta})$ is a basis of this system. It turns out that the associated Weyl group $W_{G/H}$ of the root system equals $N_H(S)/C_H(S)$. The linear automorphisms of $\Phi_{G/H}$ are precisely the automorphisms of M_T that preserve the subspace $p(M_S)$. So we conclude that the θ -invariant elements of the Weyl group W_G surjects onto $W_{G/H}$. The partition of the root system, in eqn.(15) above, produces a short exact sequence of groups

$$(17) \quad 1 \rightarrow W_L \rightarrow W_G^\theta \rightarrow W_{G/H} \rightarrow 1.$$

Let us denote $T^\theta := T \cap H$ which, under minimal rank assumption, is a maximal torus of H . The results of Brion and Joshua, see [BJ08, Section 1.4], show that the possibly non-reduced root system $\Phi_{G/H}$ is reduced, and the roots Φ_H is a subset of Φ_G . The Weyl group W_H equals the θ -invariant elements of W_G . So from the short exact sequence eqn.(17) leads to the short exact sequence

$$(18) \quad 1 \rightarrow W_L \rightarrow W_H \rightarrow W_{G/H} \rightarrow 1.$$

Let X be the wonderful compactification of a minimal rank symmetric space G/H . The closure of the torus S inside X is a toric variety, which we will denote by Y . The fan associated to this toric variety is the subdivision of N_S (the space of S -co-characters) by the Weyl chambers of the root-system $\Phi_{G/H}$. Let Y_0 denote the torus invariant affine open subset defined by the opposite Weyl chamber⁹. The Weyl group $W_{G/H}$ acts transitively on the Weyl chambers so we have $Y = W_{G/K} \cdot Y_0$. The cone corresponding to the affine subset Y_0 is of maximal dimension and we denote the unique torus fixed point associated to this cone, in Y , by z_0 .

There is a complete classification of irreducible symmetric spaces and their compactifications. In each case the topological space $PL(X)$, introduced in Section 4.3, is a graph. More precisely, all the codimension-one torus stable components are only smooth curves. A complete parametrization of the torus fixed points of X and the torus stable curves connecting these points is known.

Proposition 5.2 (See Lemma 2.1.1 [BJ08]). *We continue with the notation above. The torus fixed points and the curves of wonderful symmetric space X and the toric variety Y admit following parametrization:*

⁹This Weyl chamber corresponds to the simple roots $-\Delta_{G/H}$.

- (i) The T -fixed points of X are exactly the points $w \cdot z_0$ where $w \in W_G/W_L$. The torus fixed points of Y are exactly $w \cdot z_0$ where $w \in W_H/W_L = W_{G/H}$.
- (ii) For any positive root $\alpha \in \Phi_G^+ \setminus \Phi_L^+$, there exists unique irreducible T -stable curve $C_{\alpha \cdot z_0}$ connecting z_0 and $\alpha \cdot z_0$. The torus T acts on $C_{\alpha \cdot z_0}$ via the character α . The curve is isomorphic to \mathbb{P}^1 and we call these curves Type 1 curves.
- (iii) For any simple root $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/H}$, there exists unique irreducible T -stable curve $C_{\gamma \cdot z_0}$ connecting z_0 and $s_\alpha s_{\theta(\alpha)} \cdot z_0$. The torus T acts on $C_{\gamma \cdot z_0}$ by the character γ . The curve is isomorphic to \mathbb{P}^1 and we call these curves Type 2 curves.
- (iv) The irreducible T -stable curves in X are precisely the W_G -translates of the curves $C_{\alpha \cdot z_0}$ and $C_{\gamma \cdot z_0}$. They are all isomorphic to \mathbb{P}^1 .
- (v) The irreducible T -stable curves in Y are the $W_{G/H}$ -translates of the curves $C_{\gamma \cdot z_0}$.

5.2. Equivariant K -theory of symmetric spaces. In this section we will explore the structure of equivariant K -theory of wonderful compactifications of symmetric-spaces. Our main tools will be the previous proposition and Theorem 4.8.

5.2.1. T -equivariant K -theory. We start with an immediate consequence of Proposition 5.2.

Corollary 5.3. *The T -equivariant K -theory $K_{T,*}(X)$ of X is isomorphic to the space of tuples $(f_{w \cdot z_0}) \in \prod_{w \in W_G/W_L} K_*(k) \otimes R(T)$ such that*

$$f_{w \cdot z_0} - f_{w' \cdot z_0} = \begin{cases} 0 \mod (1 - \alpha) & \text{if } w^{-1}w' = s_\alpha \\ 0 \mod (1 - \alpha \cdot \theta(\alpha)^{-1}) & \text{if } w^{-1}w' = (s_\alpha \cdot s_{\theta(\alpha)})^\pm \end{cases}.$$

The T -equivariant K -theory $K_{T,}(Y)$ of the toric variety Y is isomorphic to the space of tuples $(f_{w \cdot z_0}) \in \prod_{w \in W_H/W_L} K_*(k) \otimes R(T)$ such that*

$$f_{w \cdot z_0} - f_{w' \cdot z_0} = 0 \mod (1 - \alpha \cdot \theta(\alpha)^{-1}) \text{ and } w^{-1}w' = (s_\alpha \cdot s_{\theta(\alpha)})^\pm.$$

Next, we relate the T -equivariant K -theories of X and Y . Let W^H denote the minimal coset representatives of W_G/W_H (recall that $W_H = W_G^\theta$). We have the following proposition.

Proposition 5.4. *There is an isomorphism of rings*

$$\prod_{w \in W^H} K_{T,*}(Y) \cong K_{T,*}(X)$$

which is compatible with the $K_(k) \otimes R(T)$ -module structure on both sides.*

Proof. Consider the chain of Coxeter groups $W_L \subset W_H \subset W_G$. It introduces a map on the quotient spaces $\pi : W_G/W_L \rightarrow W_G/W_H$. We identify W_G/W_L (resp. W_G/W_H) set-theoretically with W^L (resp. W^H) - the minimal length coset representatives. This identification defines a section, denoted by γ , to the map $\pi : W^L \rightarrow W^H$. Note that W^L is a $W_H/W_L = W_{G/H}$ torsor over W^H .

Let φ denote the composition of ring homomorphisms in eqn.(19) where the first map is canonical and the second map is the rearrangement map

$$(19) \quad K_{T,*}(X) \longrightarrow K_{T,*}(X^T) \longrightarrow \prod_{u \in W^H} \left(\prod_{v \in \pi^{-1}(u)} K_*(k) \otimes R(T) \right).$$

We denote the image of φ inside as the ring R_φ , and for any $u \in W^H$ we denote the intersection $R_\varphi \cap (\prod_{v \in \pi^{-1}(u)} K_* \otimes R(T))$ by R_φ^u . The W -action on the torus fixed points X^T translates into a W -action on the ring $\prod_{u \in W^H} (\prod_{v \in \pi^{-1}(u)} K_*(k) \otimes R(T))$ and, using Proposition 5.2 and Corollary 5.3, the action introduces ring isomorphisms

$$(20) \quad \gamma_w : R_\varphi^u \rightarrow R_\varphi^u \rightarrow R_\varphi^{u \cdot w}.$$

We use the right action here because the subgroup $W_L \subset W$ must act trivially.

The description of T -equivariant K -groups, in Corollary 5.3, show that $K_{T,*}(Y) = R_\varphi^e$ where $e \in W^H$ is the element of smallest length (i.e., it corresponds to the coset of $[H]$). So we get an inclusion map

$$(21) \quad i : K_{T,*}(Y) \rightarrow R_\varphi.$$

Given any $w \in W^H$ we define the map i_w by the composition $\gamma_w \circ i$ and this defines the map

$$i_W : \prod_{w \in W^H} K_{T,*}(Y) \rightarrow R_\varphi$$

in each co-ordinate.

The map i_W is clearly injective and it is surjective by W -equivariance. This proves the proposition. \square

5.2.2. Alternate description in terms of simplicial complex. The toric variety Y is a compactification of the maximal anisotropic torus S and as a T -variety the torus T/S acts trivially on Y . Applying Proposition 2.7 in this case, we get

$$(22) \quad K_{T,*}(Y) = K_{S,*}(Y) \otimes R(T/S).$$

In [BDCP90], the authors associate a simplicial-complex \mathcal{C}_Y to a smooth toric variety Y (see Definition 5 of [BDCP90]). The complex \mathcal{C}_Y encodes the geometry of the fan (which defines the toric variety Y). The setting in [BDCP90] is that of equivariant cohomology but their argument is geometric and it works for K -theory as well.

One can associate a Reisner-Stanley algebra to a simplicial complex, purely combinatorially over any coefficient ring, and the algebra admits a direct-sum decomposition into submodules where the summands are combinatorially defined. Using the coefficient ring $K_*(k)$ we get decomposition

$$K_{S,*}(Y) = \bigoplus_{\Delta \in \mathcal{C}_Y} K_{S,*}(Y)_\Delta$$

where the summand $K_{S,*}(Y)_\Delta$ is a $K_*(k)$ -module consists of the monomials which are supported the simplex $\Delta \subset \mathcal{C}_Y$.

This decomposition can now be exploited for the toric variety Y that appears in the description of the wonderful-compactification of the symmetric space X .

Proposition 5.5. *The T -equivariant K -theory of X admits the following direct sum- decomposition*

$$(23) \quad K_{T,*}(X) = \prod_{W^H} \left(\bigoplus_{\Delta \in \mathcal{C}_Y} K_{S,*}(Y)_\Delta \otimes R(T/S) \right).$$

where \mathcal{C}_Y is the simplex associated to the toric variety Y .

Remark 5.6. In the group case this recovers Lemma 2.8 of [Uma07].

5.2.3. *G-equivariant K-theory.* We consider the G -equivariant K -theory and we will provide two descriptions of it. The first one is a direct consequence of Proposition 5.4.

Proposition 5.7. *The G -equivariant K -theory $K_{G,*}(X)$ of X is isomorphic to W_H -invariants of the T -equivariant K -theory of the toric variety Y .*

Proof. The G -equivariant K -theory of X is given by the formula $K_{G,*}(X) = K_{T,*}(X)^{W_G}$. We have a set-theoretic splitting of $W_G = W^H \times W_H$.

It is clear that any element of $K_{T,*}(Y)^{W_H}$ embeds into $K_{G,*}(X)$. Conversely, it follows from Proposition 5.4, that after translating by an element of W^H any element of $K_{G,*}(X)$ must embed into $K_{T,*}(Y)$. Further it must also be invariant with respect to W_H -actions. So the injection $K_{T,*}(Y)^{W_H}$ is also a surjection. \square

Remark 5.8. One can also prove the above Proposition directly from the description of the torus stable curves and points on X and Y outlined in Proposition 5.2.

5.2.4. *A refined description of G-equivariant K-theory.* It turns out that one can refine Proposition 5.7 even further. The Weyl Group $W_{G/H}$ acts transitively on the Weyl chambers of the root system $\Phi_{G/H}$ and as a result the toric variety Y admits a cover by the $W_{G/H}$ -translates of an affine open set Y_0 (recall Y_0 was the affine-set corresponding to the anti-dominant Weyl chamber).

The subgroup $W_L \subset W_H$ acts trivially on Y (because it acts trivially on the fan) and hence the action of W_H on Y factors via $W_H/W_L = W_{G/H}$. The following equality then follows easily

$$K_{S,*}(Y)^{W_H} = K_{S,*}(Y)^{W_{G/H}} = K_{S,*}(Y_0).$$

This leads to the following structure theorem for G -equivariant K -theory.

Proposition 5.9. *The G -equivariant K -theory of X is*

$$K_{G,*}(X) = K_{S,*}(Y_0) \otimes R(T/S)^{W_H}.$$

Proof. In the light of Proposition 5.7, it suffices to show that

$$K_{T,*}(Y)^{W_H} = K_{S,*}(Y_0) \otimes R(T/S)^{W_H}.$$

The key-idea here is that, as a consequence of minimal-rank condition, we can identify T/S with the maximal torus of H . As a result, there is a Steinberg basis of $R(T/S)$ over $R(T/S)^{W_H}$. We will denote the elements of the Steinberg basis by $\{e_w\}_{w \in W_H}$.

It follows from eqn.(22) that $K_{S,*}(Y_0) \otimes R(T/S)^{W_H} \subset K_{T,*}(Y)^{W_H}$ and we will show the other inclusion. The Steinberg basis forms a basis $\{1 \otimes e_w\}_{w \in W_H}$ forms a basis of $K_{T,*}(Y)$ over $K_{S,*}(Y) \otimes R(T/S)^{W_H}$. As noted above W_H acts transitively on $K_{S,*}(Y)$ so taking W_H invariants we get $K_{S,*}(Y_0) \otimes R(T/S)^{W_H}$. \square

5.2.5. *Multiplicative structure constants.* As noted in Proposition 5.4, the factor $K_{T,*}(Y)$ essentially determines the T -equivariant K -theory of $K_{T,*}(X)$. The toric variety Y is determined by the reduced and irreducible root system $\Phi_{G/H}$ with Weyl group $W_{G/H}$. The root system $\Phi_{G/H}$ determines an adjoint algebraic group $\Gamma(\Phi_{G/H})$. Let $B\Gamma(\Phi_{G/H})$ denote the complete flag variety of $\Gamma(\Phi_{G/H})$. Then, using [Kly95], we can identify the toric variety Y with the closure of the general torus orbit of the maximal torus in the flag variety $B\Gamma(\Phi_{G/H})$. The following lemma is immediate.

Lemma 5.10. *Let S denote the maximal torus of $\Gamma(\Phi_{G/H})$ compatible with the root system $\Phi_{G/H}$. Then $K_{S,*}(B\Gamma(\Phi_{G/H})) = K_{S,*}(Y)$.*

The torus equivariant K -theory of flag variety has deep combinatorial structures. In particular Kostant and Kumar, in [KK90], show that the K -theory $K_{S,0}(B\Gamma(\Phi_{G/H}))$ admits a remarkable basis, called the Schubert basis, over $R(S)$. Let us denote this basis by $\{[\mathcal{O}_w]\}_{w \in W_{G/H}}$; it is indexed by the Weyl group $W_{G/H}$.

Proposition 5.11. *The torus equivariant K -theory, $K_{T,*}(X)$, admits a natural Schubert-basis over $R(T)$.*

Proof. We note that eqn.(22) expresses the fact that $K_{T,*}(X)$ is obtained by extension of scalars from $K_{S,*}(Y)$. The existence of Schubert basis for $K_{S,*}(Y)$ shows that we have a presentation

$$K_{S,*}(Y) = R(S)[\{\mathcal{O}_w\}_{w \in W_{G/H}}].$$

The proposition then follows immediately. \square

Remark 5.12. The Schubert basis (in the equivariant setting) exhibits *positivity* phenomenon (see [GK08] for precise definitions and conjectures) and it has deep combinatorial structure. The previous proposition shows that in the case of wonderful compactifications of minimal rank symmetric varieties, one can recover the structure constants from that of a lower dimensional flag-variety.

In the group case, when X is the wonderful compactification of $G \times G/\Delta(G)$, the Schubert basis $\{\mathcal{O}_w\}$ corresponds to the Schubert-basis of the flag-variety of G .

APPENDIX A. EQUIVARIANT INTERSECTION THEORY

In this section we will show that, for a G -variety X , the torus equivariant K -theory determines the torus equivariant Chow-theory. In contrast to the rest of the paper, in this section by K -theory we mean the equivariant Grothendieck group with rational coefficients, i.e, $K_{T,\mathbb{Q}}(X) := K_{T,0} \otimes \mathbb{Q}$. By Chow-theory we mean the equivariant Chow-ring with rational coefficients; as defined by Graham and Edidin. Our arguments are rather formal in nature and it works in a more general setting¹⁰. The precise hypothesis on the space X , which is always satisfied by smooth spherical projective varieties, are explained below. Our main tool is the equivariant Riemann-Roch map defined by Edidin and Graham, see [EG00].

Notation. We consider a G -variety X , and fix a maximal torus T of G with character lattice M . For example, we have $K_{T,\mathbb{Q}}(pt) := \mathbb{Q}[M]$ whereas $A_{T,\mathbb{Q}}^*(pt) := \text{Sym}_{\mathbb{Z}}(M) \otimes \mathbb{Q}$.

Let $I \subset K_{T,\mathbb{Q}}(pt)$ (resp. $J \subset A_{G,\mathbb{Q}}^*(X)$) denote the corresponding augmentation ideals. So I (resp. J) is generated by $1 - \chi$ (resp. χ) for characters $\chi \in M$. Let $\widehat{K_{T,\mathbb{Q}}(X)}$ denote the completion of $K_{T,\mathbb{Q}}(X)$ with respect to the I -adic filtration. We view $\prod_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(X)$ as a completion of the Chow-theory $\oplus_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(X)$ with respect to the sub-modules $A_{T,\mathbb{Q}}^{[n]}(X) := \prod_{i=n}^{\infty} A_{T,\mathbb{Q}}^i(X)$. The T -equivariant Riemann-Roch map, denoted by τ^T , maps $\tau^T : K_{T,\mathbb{Q}}(X) \rightarrow \prod_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(X)$ and it induces an isomorphism, also denoted by τ^T , between the completions

$$(24) \quad \tau^T : \widehat{K_{T,\mathbb{Q}}(X)} \rightarrow \prod_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(X).$$

¹⁰For example for certain non-smooth T -skeletal varieties one may substitute operational theories.

In particular, when $X = pt$, we identify $\prod_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(pt)$ with the ring of formal power-series $\sum_{i=0}^{\infty} a_i \chi^i$ (where $\chi \in M$) and the map τ^T evaluated at the element $1 - \chi$ is given by

$$(25) \quad \tau^T(1 - \chi) = \sum_{i=1}^{\infty} (-1)^{i+1} \chi^i / (i+1)!.$$

Assumption. The precise assumptions on the nature of the space X are as follows.

- We assume that the locus of T fixed points of X is finite and the natural restriction maps $K_{T,\mathbb{Q}}(X) \rightarrow K_{T,\mathbb{Q}}(X^T)$ and $A_{T,\mathbb{Q}}^*(X) \rightarrow A_{T,\mathbb{Q}}^*(X^T)$ are injective. When X is a smooth spherical G -variety these conditions are always satisfied.
- The embedding $K_{T,\mathbb{Q}}(X) \rightarrow K_{T,\mathbb{Q}}(X^T)$ is defined by finitely many congruence conditions of the form $f_i = f_j \pmod{p_{ij}}$ where f_i, f_j, p_{ij} are elements of $K_{T,\mathbb{Q}}(pt)$, and p_{ij} belongs to the augmentation ideal I .

When the equivariant K -theory (resp. Chow-theory) of X satisfies the second condition above we say that the K -theory (or Chow-theory) is *commensurable*. We will assume that the above assumptions are always satisfied.

Lemma A.1. *The following assertions are true for equivariant K -theory as well as equivariant Chow-theory.*

- The I -adic filtration on $K_{T,\mathbb{Q}}(X)$ and the filtration induced on $K_{T,\mathbb{Q}}(X)$ as a submodule of $K_{T,\mathbb{Q}}(X^T)$ with its I -adic filtration are equivalent.
- The completion map $K_{T,\mathbb{Q}}(X) \rightarrow \widehat{K_{T,\mathbb{Q}}(X)}$ is injective.

Proof. The $K_{T,\mathbb{Q}}(pt)$ module $K_{T,\mathbb{Q}}(X^T)$ is finitely generated and $K_{T,\mathbb{Q}}(pt)$ is a noetherian ring. So the first assertion is a consequence of the Artin-Rees lemma and the second assertion is a consequence of the Krull Intersection theorem (see [Mat89, Chapter 8]). The proof for equivariant Chow-theory is verbatim. \square

The following proposition is the main result of this section.

Proposition A.2. *Suppose the equivariant K -theory of X is commensurable then so is the equivariant Chow-theory of X . A commensurable presentation of the equivariant K -theory in terms of finitely many congruence conditions defines a commensurable presentation of the equivariant Chow-theory.*

Proof. Let us assume that the equivariant K -theory of X is commensurable and it is defined by finitely many congruence conditions of the form $f_i = f_j \pmod{p_{ij}}$ (where p_{ij} belongs to the augmentation ideal I) with a finite index set $(i, j) \in \mathcal{S}$.

In the Diagram (26) below, the first and the third squares commute because completion is functorial. The second square commutes because the equivariant Riemann-Roch map is functorial. The maps i (resp. j) are injective by assumption, and the maps \widehat{i} (resp. \widehat{j}) are injective by Lemma A.1.

$$(26) \quad \begin{array}{ccccccc} K_{T,\mathbb{Q}}(X) & \longrightarrow & \widehat{K_{T,\mathbb{Q}}(X)} & \xrightarrow{\tau^T} & \prod_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(X) & \longleftarrow & A_{T,\mathbb{Q}}^*(X) \\ \downarrow i & & \downarrow \widehat{i} & & \downarrow \widehat{j} & & \downarrow j \\ K_{T,\mathbb{Q}}(X^T) & \longrightarrow & \widehat{K_{T,\mathbb{Q}}(X^T)} & \xrightarrow{\tau^T} & \prod_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(X^T) & \longleftarrow & A_{T,\mathbb{Q}}^*(X^T) \end{array}$$

For any non-negative integer n , let $\mathcal{F}^n(K_{T,\mathbb{Q}}(X))$ denote the sub-module $I^n \otimes K_{T,\mathbb{Q}}(X)$ and $\widehat{\mathcal{F}^n(K_{T,\mathbb{Q}}(X))}$ the corresponding sub-module in $\widehat{K_{T,\mathbb{Q}}(X)}$. We identify the quotients $\widehat{K_{T,\mathbb{Q}}(X)}/\widehat{\mathcal{F}^n(K_{T,\mathbb{Q}}(X))}$ with $K_{T,\mathbb{Q}}(X)/\mathcal{F}^n(K_{T,\mathbb{Q}}(X))$ and similarly $\prod_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(X)/A_{T,\mathbb{Q}}^{[n+1]}(X) = \oplus_i A_{T,\mathbb{Q}}^i(X)$. By construction, the equivariant Riemann-Roch isomorphism (see eqn.(24)) is continuous with respect to the completions so we get a commutative square

$$(27) \quad \begin{array}{ccc} K_{T,\mathbb{Q}}(X)/\mathcal{F}^n(K_{T,\mathbb{Q}}(X)) & \xrightarrow{i_n} & K_{T,\mathbb{Q}}(X^T)/\mathcal{F}^n(K_{T,\mathbb{Q}}(X^T)) \\ \downarrow \tau_{n,m}^T & & \downarrow \tau_{n,m}^T \\ \oplus_{i=0}^m A_{T,\mathbb{Q}}^i(X) & \xrightarrow{j_m} & \oplus_{i=0}^m A_{T,\mathbb{Q}}^i(X^T) \end{array}$$

where $m - n$ is bounded for $m \geq n \gg 0$.

The terms in the right-hand column of the Diagram (27) are free modules, and the image of i_n stabilizes as $n \gg 0$ because the K -theory presentation is commensurable. As a result the image of j_m also stabilizes as $m \gg 0$.

The associated-graded ring of $\prod_{i=0}^{\infty} A_{T,\mathbb{Q}}^i(X^T)$ (with respect to the filtration $A_{T,\mathbb{Q}}^{[n]}(X^T)$) is given by $A_{T,\mathbb{Q}}^*(X^T)$. As a result we note that the image of $A_{T,\mathbb{Q}}^*(X)$ is determined inside $A_{T,\mathbb{Q}}^*(X)$ by finitely many relations. Moreover, passing to the associated graded ring the image of the finitely many relations

$$\{\tau_{n,m}^T(f_i) = \tau_{n,m}^T(f_j) \mod \tau_{n,m}^T(p_{ij})\}_{(i,j) \in \mathcal{S}}$$

determine a commensurable presentation of the Chow-ring. □

Remark A.3. We can recover Brion's calculation of equivariant Chow-theory of smooth projective spherical variety using Theorem 4.8 and the above proposition.

A similar result also holds for G -equivariant theories because the G -equivariant theory is determined by the invariants of the geometric action of the Weyl-group on the torus fixed points.

REFERENCES

- [Ahi83] Dmitry Ahiezer. Equivariant completions of homogeneous algebraic varieties by homogeneous divisors. *Ann. Global Anal. Geom.*, 1(1):49–78, 1983.
- [AHW09] Suanne Au, Mu-wan Huang, and Mark E. Walker. The equivariant -theory of toric varieties. *Journal of Pure and Applied Algebra*, 213(5):840 – 845, 2009.
- [AP15] Dave Anderson and Sam Payne. Operational K -theory. *Doc. Math.*, 20:357–399, 2015.
- [BBJ16] S. D. Banerjee, M. Bilen Can, and M. Joyce. Combinatorial Models for the Variety of Complete Quadrics. *ArXiv e-prints*, October 2016.
- [BDGP90] Emili Bifet, Corrado De Concini, and Claudio Procesi. Cohomology of regular embeddings. *Adv. Math.*, 82(1):1–34, 1990.
- [BJ08] Michel Brion and Roy Joshua. Equivariant Chow ring and Chern classes of wonderful symmetric varieties of minimal rank. *Transform. Groups*, 13(3-4):471–493, 2008.
- [BLR12] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21. Springer Science & Business Media, 2012.
- [Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
- [Bri86] Michel Brion. Quelques propriétés des espaces homogènes sphériques. *Manuscripta Math.*, 55(2):191–198, 1986.

- [Bri87] M. Brion. Classification des espaces homogènes sphériques. *Compositio Mathematica*, 63(2):189–208, 1987.
- [Bri97] Michel Brion. Equivariant Chow groups for torus actions. *Transform. Groups*, 2(3):225–267, 1997.
- [Car94] James B. Carrell. The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties. In *Algebraic groups and their generalizations: classical methods (University Park, PA, 1991)*, volume 56 of *Proc. Sympos. Pure Math.*, pages 53–61. Amer. Math. Soc., Providence, RI, 1994.
- [DCP83] Corrado De Concini and Claudio Procesi. Complete symmetric varieties. In *Invariant theory (Montecatini, 1982)*, volume 996 of *Lecture Notes in Math.*, pages 1–44. Springer, Berlin, 1983.
- [DCS85] C. De Concini and T. A. Springer. Betti numbers of complete symmetric varieties. In *Geometry today (Rome, 1984)*, volume 60 of *Progr. Math.*, pages 87–107. Birkhäuser Boston, Boston, MA, 1985.
- [DCS99] C. De Concini and T.A. Springer. Compactification of symmetric varieties. *Transformation Groups*, 4(2-3):273–300, 1999.
- [EG00] Dan Edidin and William Graham. Riemann-roch for equivariant chow groups. *Duke Math. J.*, 102(3):567–594, 05 2000.
- [FG05] Eric M. Friedlander and Daniel R. Grayson, editors. *Handbook of K-theory. Vol. 1, 2*. Springer-Verlag, Berlin, 2005.
- [Fog73] John Fogarty. Fixed point schemes. *Amer. J. Math.*, 95:35–51, 1973.
- [Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.
- [GK08] William Graham and Shrawan Kumar. On positivity in T -equivariant K -theory of flag varieties. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 093, 43, 2008.
- [GKM98] Mark Goresky, Robert Kottwitz, and Robert MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. *Invent. Math.*, 131(1):25–83, 1998.
- [Gon15] Richard P. Gonzales. Equivariant operational Chow rings of T -linear schemes. *Doc. Math.*, 20:401–432, 2015.
- [JK15] Roy Joshua and Amalendu Krishna. Higher K -theory of toric stacks. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, XIV(5):1–41, 2015.
- [KK90] Bertram Kostant and Shrawan Kumar. T -equivariant K -theory of generalized flag varieties. *J. Differential Geom.*, 32(2):549–603, 1990.
- [Kly95] A. A. Klyachko. Toric varieties and flag spaces. *Trudy Mat. Inst. Steklov.*, 208(Teor. Chisel, Algebra i Algebr. Geom.):139–162, 1995. Dedicated to Academician Igor’ Rostislavovich Shafarevich on the occasion of his seventieth birthday (Russian).
- [Mat89] H. Matsumura. *Commutative Ring Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1989.
- [Mer97] Alexander S. Merkurjev. Comparison of the equivariant and the standard K -theory of algebraic varieties. *Algebra i Analiz*, 9(4):175–214, 1997.
- [Ric82] R.W. Richardson. Orbits, invariants, and representations associated to involutions of reductive groups. *Inventiones mathematicae*, 66(2):287–312, 1982.
- [Spr] TA Springer. Some results on algebraic groups with involutions. algebraic groups and related topics (kyoto/nagoya, 1983), 525–543. *Adv. Stud. Pure Math*, 6.
- [Spr09] Tonny A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, second edition, 2009.
- [Ste75] Robert Steinberg. On a theorem of Pittie. *Topology*, 14:173–177, 1975.
- [Str92] Elisabetta Strickland. Equivariant Betti numbers for symmetric varieties. *J. Algebra*, 145(1):120–127, 1992.
- [Str12] Elisabetta Strickland. The Reisner-Stanley system and equivariant cohomology for a class of wonderful varieties. *J. Algebra*, 356:216–229, 2012.
- [Tch07] Alexis Tchoudjem. Cohomologie des fibrés en droites sur les variétés magnifiques de rang minimal. *Bull. Soc. Math. France*, 135(2):171–214, 2007.
- [Tho85] Robert W. Thomason. Algebraic K -theory and étale cohomology. *Annales scientifiques de l’École Normale Supérieure*, 18(3):437–552, 1985.

- [Tho87] Robert W. Thomason. Algebraic K -theory of group scheme actions. In *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*, volume 113 of *Ann. of Math. Stud.*, pages 539–563. Princeton Univ. Press, Princeton, NJ, 1987.
- [Uma07] Vikraman Uma. Equivariant K -theory of compactifications of algebraic groups. *Transform. Groups*, 12(2):371–406, 2007.
- [VV02] Gabriele Vezzosi and Angelo Vistoli. Higher algebraic K -theory of group actions with finite stabilizers. *Duke Math. J.*, 113(1):1–55, 2002.
- [VV03] Gabriele Vezzosi and Angelo Vistoli. Higher algebraic K -theory for actions of diagonalizable groups. *Invent. Math.*, 153(1):1–44, 2003.
- [Wat79] William C. Waterhouse. *Introduction to affine group schemes*, volume 66 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979.